

Sheet 1. Introduction to Dynamical Systems

The Birthday Problem: Write down the day of the month you were born on. If you were born on January 24th, you would write 24.

- ◊ Take that number, multiply it by 1/2 and add 4 to the result. If you started with 24, you put down 16.
- ◊ Take the number you got, multiply it by 1/2 and add 4 to the result.
- ◊ Take the number you got, multiply it by 1/2 and add 4 to the result, etc.

If you started with 24, you would have created the sequence

24, 16, 12, 10, 9, 8.5, 8.25, 8.125, 8.0625…

The field of mathematics which studies sequence of numbers created in this way is called *Dynamical Systems*. The number you start with is the *seed*, the sequence is called an *orbit* and each term in the sequence is called an *iterate* (because it is the result of the *iteration* of a function.)

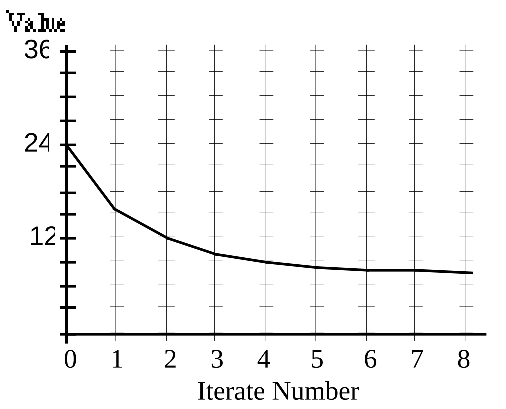
1. Fill in a *times series table* for the rule $y = (1/2)x + 4$ using seeds of 24, 18, 12, 6, 0, and two of your own choosing. Examine the table carefully and list any patterns you observe.

Time Series Table

Seed	Iterate Number							
0	1	2	3	4	5	6	7	8
24	16	12	10	9	8.5	8.25	8.125	8.0625

$y = .5x + 4$

Time Series Graph



2. Fill in a times series table for the rule $y = (3/2)x - 2$ using seeds of -1, 2, 5, 8 and three of your own choosing. Examine the table carefully and list below any patterns you observe.

A *time series graph* will give you another way to look at orbits. Using the function $y = (1/2)x + 4$, and the seed 24, form a set of ordered pairs for the orbit calculated above, where the first number is the iterate number and the second number is the iterate value.

(0, 24), (1,16), (2,12), (3, 10), (4,9) (5, 8.5), (6, 8.25), (7, 8.125) , (8, 8.0625)

These points have been plotted on the Time Series Graph above, and each point was joined to its successor with a line segment.

3. Plot the time series graph for the orbits you calculated in #1. Examine the graph. What patterns do you observe? How do they relate to what you observed in #1?

4. Plot the time series graph for the orbits you calculated in #2. Examine the graph. What patterns do you observe? How do they relate to what you observed in #2?

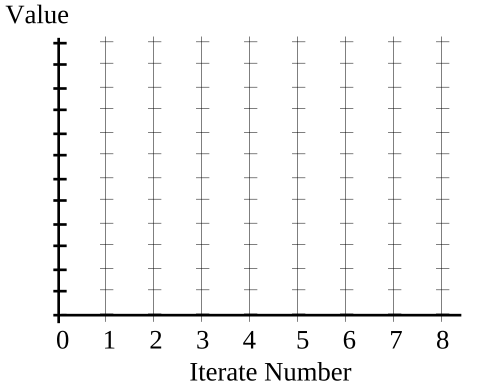
5. For each of the two equations, find the point whose time series graph would be a horizontal line. Explain how you found it. Do the time series graphs for other seeds intersect this one?

Time Series Table

Seed	Iterate Number							
0	1	2	3	4	5	6	7	8

y =

Time Series Graph

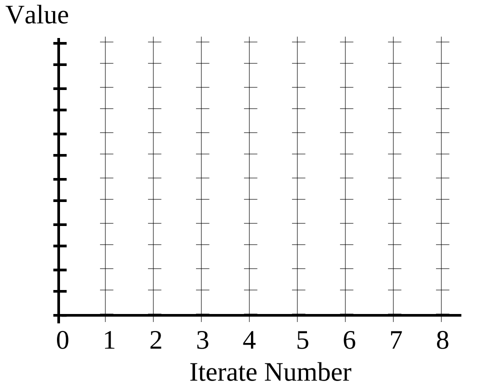


Time Series Table

Seed	Iterate Number							
0	1	2	3	4	5	6	7	8

y =

Time Series Graph

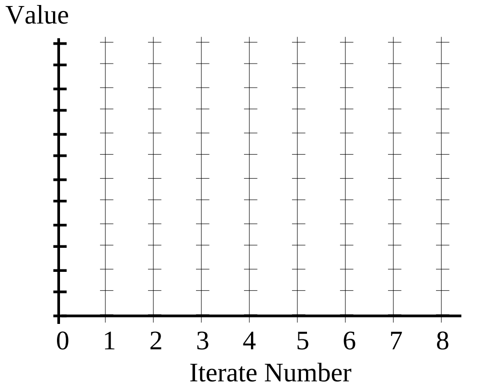


Time Series Table

Seed	Iterate Number							
0	1	2	3	4	5	6	7	8

y =

Time Series Graph

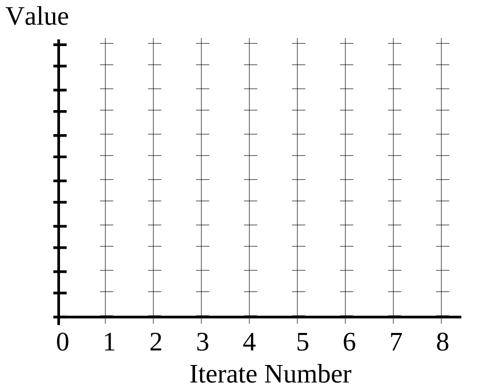


Time Series Table

Seed	Iterate Number							
0	1	2	3	4	5	6	7	8

y =

Time Series Graph



Sheet 2: Modeling Medication

Dynamical systems can be used to model how your body deals with medications such as cold remedies. Once you ingest some medication, your body eliminates it from your system in such a way that a fixed percentage of the amount remaining in your system is removed per time period.

In what follows you will study how your body eliminates an imaginary medication called FluRidder. Assume that:

- Your body eliminates 32% of the FluRidder that is in your system per hour.
- You have taken 100 units of FluRidder initially.
- You take an additional hourly dose of 40 units beginning one hour after you took the initial dose.

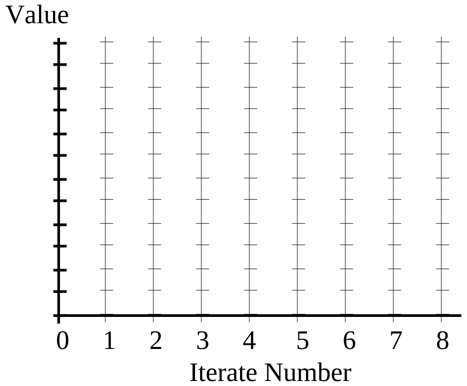
1. a. What percent of the FluRidder is *not* eliminated at the end of an hour?
   b. If you had x units of FluRidder at the beginning of an hour, how much would you have at the end?

2. Explain why you could iterate the function y = x - 0.32x + 40 or y = (0.68)x + 40 to model the amount of FluRidder in your system. What does x stand for? What does the 0.68x stand for? What does the 40 stand for? What does y stand for?

3. Make a time series table and graph using initial doses of 40, 80, 120, and 160 and three of your own choice, which shows the amount of medication in your system over an 8-hour period.

Time Series Table								
Seed	Iterate Number							
0	1	2	3	4	5	6	7	8
y =								

Time Series Graph



4. Suppose you need at least 80 units of FluRidder in your system for it to be effective. If you were to take an initial dose of 20 units instead of 80 units, how long before you would be feeling the effects of FluRidder?

5. Suppose you were an Olympic athlete and that you have been taking FluRidder to help you recover from a cold. You have been taking it for three days, starting each day with an initial dose of 100 units at 6 a.m., and taking hourly doses of 40 units until 10 p.m. You learn at 1:30 p.m. on Monday that FluRidder is on the list of banned substances and that you will have to take a test which will detect .01 units of the drug. If you stop taking the medication, when is the earliest time you could take the test and pass it?

6. You could use the function y = 0.68(x + 40) to model this situation. How does it differ from the one in #2 in terms of the real life situation? (Hint: which operation is done first in each version of the formula?)

Sheet 3: Financial Mathematics

Dynamical systems can also be used to model how financial institutions maintain different types of customer accounts. In what follows assume

- The bank pays you 1% interest per month.
- At the end of the first month, you begin making regular monthly deposits of $75.

1. Explain why you would iterate the function y = x + 0.01x + 75 = 1.01x + 75 to model the growth of your balance in the account. What does the x stand for? What does the .01x stand for? What does the 75 stand for? What does the y stand for?

2. Suppose you were saving up to purchase a racing bicycle which costs $985. How long will it take you to have that much money in your account if you started with $100 in the account, and made your first deposit at the end of one month?

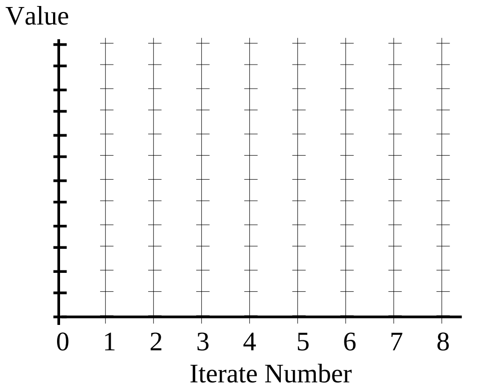
3. Suppose that instead of waiting for your savings to grow, you borrowed the $885 you needed (on top of the $100 you had) using the same type of account (1% monthly interest and monthly payments of $75). Explain why you can think of this situation as having a negative balance in the account. You have to keep making monthly payments of $75 until your balance is $0. How long would it take you to pay off the loan? How much would the bike actually end up costing you?

4. Make a times series graph for y = 1.01x + 75.

Time Series Table

Seed	Iterate Number							
0	1	2	3	4	5	6	7	8
y =								

Time Series Graph



5. Many states have what are called Megabucks lotteries. The payoffs in these lotteries are usually $1,000,000 but you don’t get the money right away. The way most states pay off the winner is to pay $50,000 right away and then make $50,000 payments for the next 19 years. In order to finance the payments the bank places an initial deposit in an account that pays interest, makes annual payments of 50,000 and has a final balance of $0 after the 19th payment is made.
   a. Suppose the state uses an account that pays 10% annual interest (interest is paid once a year at a rate of 10% per year), explain why the function y = 1.1x - 50,000 models this situation.
   b. Find the initial deposit.

6. You could have used the function y = 1.01(x + 75). How does this formula differ from the one in #1 in terms of the real life situation?

Sheet 4: Fixed Points

To work on this sheet, you will need to refer back to your time series graphs from the previous sheets.

1. a. What should the initial dose of FluRidder be, if you want the amount in your system to be back to the same level after you take your first hourly dose one hour later? (Your body eliminates 32% of the FluRidder in your system, and you take a 40 unit hourly dose. See Sheet 2.)
   b. Someone with shaky math understanding takes out a loan at 1% monthly interest, and makes monthly payments of $75. (See Sheet 3.) However he finds that his balance due remains constant, month after month. What was the amount of the loan?

These are examples of *fixed points.* A fixed point for a function is the value of x for which y = x. For example, in the birthday problem (see Sheet 1), if you start with a seed of 8, and iterate the function y = 0.5x + 4, all the iterates equal 8.

- ◊ To check if a number is a fixed point for a function, you just substitute it for x, and see if you get an equal value for y.
- ◊ To find the fixed point for y = mx + b, you can solve the equation x = mx + b.

2. What does the orbit of a fixed point look like on a time series graph?

3. Find the fixed points for these functions:
   a. y = .9x + 10 b. y = 2.8x - 12.4

4. a. Use algebra to find a formula for the fixed point of the function y = mx + b, by solving the equation x = mx + b for x.
   b. Use this formula to check the fixed points you found in #1.

As you probably noticed, all orbits in the birthday problem get closer and closer to the fixed point (8). Similarly, all orbits in the FluRidder problem get closer and closer to the fixed point (125). The same thing did not happen with the financial problem, even though there was a fixed point.

5. What happens to the orbits of seeds that are close to 125 when you iterate y = (0.68)x + 40? Be sure to try both larger and smaller values. Explain why 125 is called an *attracting fixed point.*

6. What happens to the orbits of seeds that are close to -7500 when you iterate y = (1.01)x + 75? Be sure to try both larger and smaller values. Explain why -7500 is called a *repelling fixed point*.

To understand this, look at the time series graphs you made in Sheets 1-3. In some graphs, the lines move closer to each other (and to the fixed point's horizontal line) as you move from left to right. In other graphs, the lines move away from each other (and from the fixed point's horizontal line.)

7. What feature of the equation seems to determine whether the lines move towards each other or away from each other? If necessary, make additional time series graphs to check your conjecture.

8. These are questions about the equations in #3:
   a. Predict which one would yield a time series graph where the lines move away from each other, and which a graph where the lines move towards each other.
   b. Predict which one has an attracting fixed point, and which a repelling fixed point.
   c. Check your predictions.

9. Write a summary of everything you know about iterating linear functions, and their applications to real life problems. Be sure to mention fixed points and the role of the parameter m in y = mx + b, to discuss examples you make up, and to illustrate them with time series tables and graphs. Your summary will be evaluated based on:
   - ◊ Meaningful and mathematically correct examples
   - ◊ Correct use of terminology
   - ◊ Correct use of the time series table and graph
   - ◊ Clarity and neatness