Jonathan Choate and Henri Picciotto

Iterating Linear Functions—an Introduction to Dynamical Systems

TEACHER'S GUIDE

The arrival of computers has caused some major changes in both mathematics and mathematics education. One of the biggest shifts has been from an emphasis on symbolic methods to one on numerical methods. One field of mathematics, dynamical systems, requires considerable number crunching and is just coming into its own because we currently have the ability to perform extensive calculations easily. This article introduces students to this new field. The study of sequences created by using numerical iteration provides interesting new ways to approach many of the concepts central to the secondary mathematics curriculum, such as functions in general and linear and exponential functions in particular. Although the activities are calculator dependent, students will still have to reason graphically, numerically, and algebraically about functions. They will have to interpret graphs, find patterns in tables of data, and use algebra both to simplify expressions and to derive new meanings from them. In addition, they will be given some new modeling tools and will be using what they learn in an applications setting. Finally, these activities will give students the necessary background to explore later the exciting new mathematics of chaos and fractal geometry. Most of the study of dynamical systems is about nonlinear functions. The linear cases explored in this activity build a foundation for such later advanced studies.

Prerequisites: Basic algebra

Grade level: 9–12

Materials: The activities require a lot of computation and can be done using any calculator, but one with an **ANS** key is preferable. A spreadsheet program can also be used.

Objectives: These activities will enable students to learn the basics of numerical iteration and then use it as a tool for creating mathematical models.

Sheets 1A and 1B: These sheets introduce the basic concepts, vocabulary, and tools for this activity. Use the birthday problem to establish how students will perform the calculations, depending on what technology they can use. If they have calculators with an **ANS** key, the following explains how to generate orbits, using the example given on the student sheet:

- Clear the screen
- Enter 24 in the display and press **ENTER**.
- Type in (1/2)∗ANS + 4 and press **ENTER**; 16 should appear in the display.
- Press **ENTER** again, and 12 will appear.
- Each time **ENTER** is pressed, the next term in the sequence will appear.

Other methods are possible, depending on the capabilities of the calculators or software that your students are using. Ask students to compare their results with those of their neighbors. They should find that all orbits get closer and closer to 8. If some students do not, help them check their calculation technique.

Demonstrate how to fill out the table and graph for numbers 1 and 3 on sheets 1A and 1B. Note that students can share the computation with their

Jonathan Choate teaches mathematics at Groton School, Groton, MA 01450. He is interested in the mathematics of fractals, chaos, and complexity and how they can be used. Henri Picciotto has taught everything from counting to calculus. He is involved in teacher training and curriculum development and is interested in manipulative and electronic environments.

Edited by ***Henri Picciotto,*** *hot@soe.berkeley.edu, The Urban School, San Francisco, CA 94117*

This section is designed to provide in reproducible formats mathematics activities appropriate for students in grades 7–12. This material may be reproduced by classroom teachers for use in their own classes. Readers who have developed successful classroom activities are encouraged to submit manuscripts, in a format similar to the "Activities" already published, to the senior journal editor for review. Of particular interest are activities focusing on the Council's curriculum standards, its expanded concept of basic skills, problem solving and applications, and the uses of calculators and computers.

Write to NCTM, Department P, or send e-mail to infocentral@nctm.org, for the catalog of educational materials, which lists compilations of "Activities" in bound form.—Ed.

neighbors but that everyone must end with a complete record. For numbers 2 and 4, students will complete another time-series table and graph. They will have to decide how to set up the scale for the graph, which is easier to do after having computed a few orbits.

The time-series graph showing several orbits at once is an application and refinement of the function diagram and linked f_n diagram representations (Wah and Picciotto 1994).

Some students may notice that the time-series graphs look like exponential curves, and indeed they are. See the Mathematical Background section for an explanation.

For number 5, accept any reasonable answer, including a visual one based on looking at the graphs and guessing. However, make sure that students check that the iterates for the seed they suggest as having a horizontal graph indeed remain fixed. Some students may notice that in the first example, the time-series graphs all seem to get closer and closer to the fixed point, whereas in the second example, they seem to move away from it. They will be asked to do a more thorough analysis of the fixed point on sheets 4A and 4B.

Sheet 2: To verify an understanding of numbers 1 and 2, discuss what the formula would be with different values for the percent eliminated per hour or for the hourly dose. Also ask the students why the initial dose of 100 units does not show up in the formula. (Answer: It is actually a seed, not a part of the equation.)

The answer to number 4 depends on how you interpret the question. Does the level of FluRidder need to pass 80 units some of the time or does it need never to drop below 80 units? When going over this situation with your students, make a table that shows the level of medicine both immediately before and immediately after the hourly dose is taken. Here is a detailed analysis of the problem.

The level of FluRidder in the body immediately after the first dose is taken is 20 units. Then the level drops gradually until it is down to $0.68 \cdot 20 = 13.6$ units immediately before the next dose is taken. Immediately after that dose is taken, the level is $13.6 + 40 = 53.6$ units, which is the number given by the formula. After two more iterations, the formula gives 91.99 units, which is greater than 80 and satisfies the first condition. Three hours have passed.

However, at some point during the following hour, the level will drop below 80 units on its way to a low of $0.68 \cdot 91.99 = 62.55$ units. To make sure that the level never drops below 80 units, you must wait for the amount in your system to reach or pass 117.65 units right after you take the medicine, which would guarantee that you had at least 80 units in your system right before taking the hourly dose, since $0.68 \cdot 117.65 = 80.00$. This level occurs at the end of seven hours when the amount of FluRidder in the system is 117.94 units.

Note that an alternative formula to use in thinking about this problem is $y = 0.68(x + 40)$. In this case, the x represents the amount of medicine in the system immediately before the hourly dose is taken, and the y, the amount immediately before the next dose is taken. The seed for number 4 would be 13.6 units, the amount immediately before the first 40-unit dose is taken. If you iterate this formula, you will find that the amount of medication in your system before taking the hourly dose stabilizes at about 85 units. In contrast, iterating the formula from number 2 shows that the amount of FluRidder in the system immediately after the hourly dose is taken stabilizes at about 125 units. Not surprisingly, this level is 40 units higher. Note that 40 is 32 percent of 125, as it should be.

It is not necessary for students to fully understand this concept right away. Depending on the class, you can have a full discussion of the question, analyzing what happens with each formula, or a brief discussion of the situation being modeled, stressing what part of it is represented by the formula in number 2.

Since most of the FluRidder is purged from your body during the night, it turns out that you get the same answer whether you do number 5 by carrying out the iteration over three days or by starting at 6:00 A.M. that day. Again, you should decide how deeply to get into this analysis on the basis of the level of the class.

Sheet 3: These applications to financial matters work the same way, although they may be a little easier to understand, since no continuous change is involved as in the medication problem. However, students may need help figuring out how to use negative numbers to describe the loan in number 3.

In number 4, students need to choose both the seeds and the scale of the axis. It is imperative that the range from –1000 to 1000 be displayed, since it is what the bicycle problem requires. However, for students to find the horizontal line in the graph, the lower bound of the scale needs to include –7500 (see sheet 4A, 2).

Number 5 is challenging. Trial and error can quickly narrow the search to an approximate range, but to get an exact answer, it is best to work backward: if the amount in the account ends at \$0, it had \$50 000 in it immediately preceding the last payment, and it had \$50 000/1.1 one year before that. More generally, if x is in the account at a given time, a year earlier the amount was $y = (x + 50\,000)/1.1$. This situation is the inverse function to the one mentioned in the problem. To find the initial deposit, iterate this function nineteen times, with a seed of 0.

The general linear function that gets iterated for many financial accounts is $y = (1 + r)x + P$, where r is the interest rate and P is the amount of money deposited or withdrawn. With a seed of S, here is a summary of the types of accounts:

- If $S > 0$ and $P = 0$, then the account is a *savings account*.
- If $S > 0$ and $P > 0$, then the account is an *annuity* or *retirement fund*.
- If $S > 0$ and $P < 0$, then the account is a *sinking fund*.
- If $S < 0$ and $P > 0$, then the account is a *loan* or a *mortgage*.

Sheet 4A: If number 5*(a)* is too difficult for your class, you may demonstrate the solution. Students at all levels should see that the value of m in $f(x) = mx + b$ determines the type of fixed point. It would be appropriate for upper-level students to prove their conjectures, which can be done in a variety of ways, one of which is outlined in the mathematical-background section that follows.

One way to test whether the lines of the time-series graphs move toward or away from one another is to test the first iterates of seeds 0 and 1: $y = m(0) + b = b$ and $y = m(1) + b = m + b$. The difference between the seeds was 1, and the difference between the first iterates is m. It follows that if $m > 1$, the lines move away from one another (repelling fixed point), and if $0 < m < 1$, the lines move toward one another (attracting fixed point). More generally, if you take two seeds x_a and x_b and the corresponding first iterates y_a and y_b, the ratio $(y_b - y_a)/(x_b - x_a)$ indicates whether the lines move away from or toward one another. Of course, that ratio is m, the rate of change of the function, which we usually think of as the slope of the line in a Cartesian graph.

MATHEMATICAL BACKGROUND

If you are not familiar with this material, start by working through the student sheets. Then for more information, read through this section. To allow access to the student sheets for the broadest possible range of classes, we avoided function and subscript notation. However, we use both in this section.

We label the seed x_0, and say that the sequence $x_0, x_1, x_2, x_3, \ldots, x_k$ is the orbit of x_0 under iteration of $f(x)$. Note that $x_1 = f(x_0)$, $x_2 = f(x_1)$, and so on, and that in general $x_k = f(x_{k-1})$. In other words, the next number in the orbit is calculated by applying the function rule to the preceding number. *Time-series graphs* are produced by plotting the points $(0, x_0)$, $(1, x_1)$, $(2, x_2)$, ... and then connecting each point to its predecessor with a line segment. The line segments are not really part of the graph but show how the points are related to one another.

By definition, $f(x)$ has a fixed point at $x = F$ if $f(F) = F$. By solving the equation $x = mx + b$ for x, you can show that the linear function $f(x) = mx + b$ has a fixed point at $x = F = b/(1 - m)$. Once you have found the fixed point for a function, the next question of interest is what kind of fixed point is it, attracting or repelling? A little algebra throws light on this question. Since $b = (1 - m)F$, the function can be written $f(x) = mx + F - Fm$, or $f(x) - F = m(x - F)$. Therefore, the relationship between consecutive iterates is $x_k - F = m(x_{k-1} - F)$, meaning that the directed distance between an iterate and the fixed point equals m times the directed distance between the previous iterate and the fixed point. This relationship has interesting consequences:

- The distance between iterates and the fixed point is a geometric sequence, which explains the exponential shape of the time-series graphs.
- If $|m| < 1$, the distance decreases, we have an attracting fixed point, and the orbits decay toward it exponentially.
- If $|m| > 1$, the distance increases, we have a repelling fixed point, and the orbits grow away from it exponentially.

ASSESSMENT

Number 10 on sheet 4B provides an opportunity to evaluate students' understanding. Expect a two-page illustrated summary of the key ideas, including examples. You may offer extra credit for a clear explanation of the "inverse" problem (see the lottery problem: sheet 3, 5).

EXTENSIONS

Algebraic analysis

If students have studied sequences and series and the composition of functions, they can explore the ideas in the mathematical-background section. An alternative approach is to observe that one can iterate $f(x) = mx + b$ in the general case and produce the sequence x, $mx + b$, $m(mx + b) + b = m^2x + mb + b$, ... Using the formula for the sums of a finite geometric series, students can derive a formula for the kth iterate and use it to analyze the behavior of the orbits.

More on dynamical systems

For more on dynamical systems, you and your students can look at Devaney (1989), McGuire (1990), Sandefur (1993), and Heid and others (1995). The TI-82, TI-83, and TI-92 calculators have a built-in Sequence mode that allows the user to generate orbits quickly in both table forms. Moreover, they support so-called "web" diagrams of the iteration process. These features make it possible to explore the iteration of nonlinear functions, such as quadratic and trigonometric functions.

Selected answers: Sheets 1A and 1B: 1 and 3. All graphs and iterate values head toward $y = 8$. See **figure 1.**

2 and 4. All graphs and iterate values except $y = 4$ move further and further away from $y = 4$. See **figure 2.**

5. First graph: the horizontal line for the seed 8. Second graph: the horizontal line for the seed 4.

Sheet 2: 1. *(a)* 68 percent and *(b)* $0.68x$

2. The quantity x is the amount of FluRidder in your body at the very beginning of an hour. The amount $0.68x$ is left in the body an hour later, 40 is the number of units in the hourly dose, and y is the amount of medication in your body at the very end of an hour right after you take the hourly dose.

3. See **figure 3.**

4. 3 hours to reach 80 units and 7 hours to never drop below 80 units. See the Teacher's Guide.

5. 2:00 P.M. on Tuesday

Sheet 3: 1. The x stands for the amount of money in the account at the beginning of a given month, $0.01x$ is the interest earned by the end of the month, and 75 is the dollar amount of the monthly deposit. The y stands for the amount of money in the account at the end of a month right after the monthly deposit is made.

2. 12 months

3. 13 months. The total cost would be $100 + 13 \cdot 75$ minus the amount of money in the account at that time, which is \$28.49, or \$1046.51.

4. Answers will vary.

5. *(a)* $1.1x$ represents the amount in the account after the interest has been received. The –50 000 is the amount paid to the winner each year.
(b) \$418 246.01. See the Teacher's Guide.

Sheets 4A and 4B: 1. 125 units

2. \$7500

3. A horizontal line

4. *(a)* 100 and *(b)* 6.8888 . . .

5. *(a)* $-b/(m-1)$

6. The orbits approach 125.

7. The orbits move away from –7500.

8. See the Teachers' Guide.

Time-Series Table

Seed	Iterate Number 0	1	2	3	4	5	6	7	8
24	24	16	12	10	9	8.5	8.25	8.13	8.06
18	18	13	10.5	9.25	8.63	8.31	8.16	8.08	8.04
12	12	10	9	8.5	8.25	8.13	8.06	8.03	8.02
6	6	7	7.5	7.75	7.88	7.94	7.97	7.98	7.99
0	0	4	6	7	7.5	7.75	7.88	7.94	7.97

$y = 0.5x + 4$

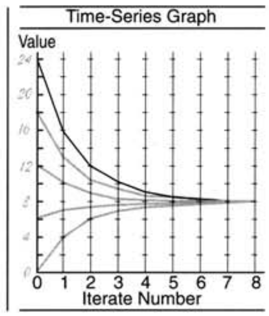


Fig. 1

Time-Series Table

Seed	Iterate Number 0	1	2	3	4	5	6	7	8
–1	-1	-3.5	-7.25	-12.88	-21.31	-33.97	-52.95	-81.43	-124.1
2	2	1	-0.5	-2.75	-6.13	-11.19	-18.78	-30.17	-47.26
5	5	5.5	6.25	7.38	9.06	11.59	15.39	21.09	29.63
8	8	10	13	17.5	24.25	34.38	49.56	72.34	106.5

$y = 1.5x - 2$

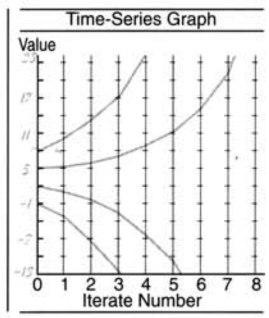


Fig. 2

Time-Series Table

Seed	Iterate Number 0	1	2	3	4	5	6	7	8
40	40	67.20	85.70	98.27	106.83	112.64	116.60	119.29	121.11
80	80	94.40	104.19	110.85	115.38	118.46	120.55	121.97	122.94
120	120	121.60	122.69	123.43	123.93	124.27	124.51	124.66	124.77
160	160	148.80	141.18	136.01	132.48	130.09	128.46	127.35	126.60

$y = 0.68x + 40$

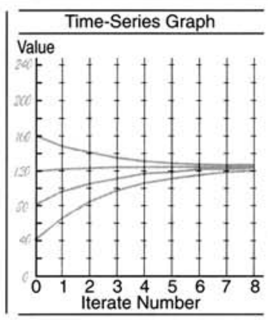


Fig. 3

9. $y = 0.9x + 10$ has lines that move toward one another and an attracting fixed point; $y = 2.8x - 12.4$ has lines that move away from one another and a repelling fixed point.

Devaney, Robert. *Chaos, Fractals and Dynamics: Computer Experiments in Mathematics*. Reading, Mass.: Addison-Wesley Publishing Co., 1989.

Heid, M. Kathleen, Jonathan Choate, Charlene Sheets, and Rose Mary Zbiek. *Algebra in a Technological World.* Grades 9–12 Addenda series. Reston, Va.: National Council of Teachers of Mathematics, 1995.

McGuire, Michael. *An Eye for Fractals: A Graphic-Photographic Essay*. Reading, Mass.: Addison-Wesley Publishing Co., 1995.

Sandefur, James. *Discrete Dynamical Modeling.* New York: Oxford University Press, 1993.

Wah, Anita, and Henri Picciotto. *Algebra: Themes, Tools, and Concepts.* Mountain View, Calif.: Creative Publications, 1994.

(Worksheets begin on page 126)

The Birthday Problem: Write down the day of the month on which you were born. If you were born on 24 January, you would write "24."

- Take that number, multiply it by 1/2, and add 4 to the result. If you started with 24, you put down 16.
- Take the number you got, multiply it by 1/2, and add 4 to the result.
- Take the number you got, multiply it by 1/2, add 4 to the result, and so on.

If you started with 24, you would have created the sequence 24, 16, 12, 10, 9, 8.5, 8.25, 8.125, 8.0625 . . . .

The field of mathematics that studies sequences of numbers created in this way is called *dynamical systems.* The number you start with is the *seed;* the sequence is called an *orbit;* and each term in the sequence is called an *iterate* because it is the result of the *iteration* of a function.

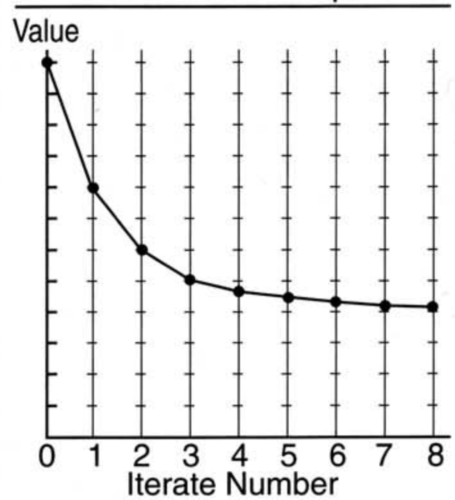
1. Fill in a *times-series table* for the rule $y = (1/2)x + 4$ using the seeds 24, 18, 12, 6, 0 and two of your own choosing. Round your answers to two decimal places. Examine the table carefully and list on a separate sheet any patterns that you observe.

Time-Series Table

Seed	Iterate Number								
	0	1	2	3	4	5	6	7	8
24	24	16	12	10	9	8.5	8.25	8.13	8.06
18									
12									
6									
0									

$y = 0.5x + 4$

Time-Series Graph



From the *Mathematics Teacher,* February 1997

(Continued on page 132)

(Continued from page 126)

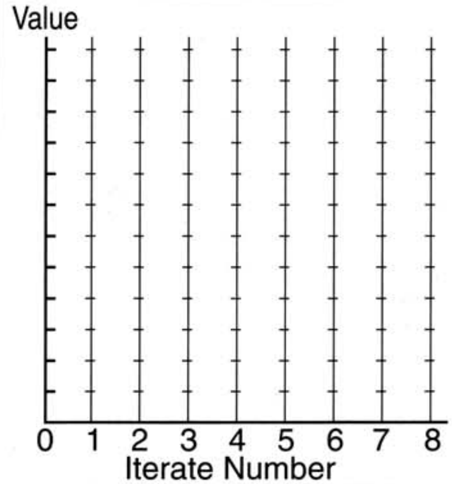
2. Fill in a times-series table for the rule $y = (3/2)x - 2$ using the seeds −1, 2, 5, 8, and three of your own choosing. Examine the table carefully and list on a separate sheet any patterns that you observe.

Time-Series Table

Seed	Iterate Number								
	0	1	2	3	4	5	6	7	8
−1									
2									
5									
8									

$y = 1.5x - 2$

Time-Series Graph



A *time-series graph* will give you another way to look at orbits. From function $y = (1/2)x + 4$ and the seed 24, a set of ordered pairs follows for the orbit previously calculated. The first number is the iterate number, and the second number is the iterate value.

(0, 24), (1, 16), (2, 12), (3, 10), (4, 9), (5, 8.5), (6, 8.25), (7, 8.13), (8, 8.06)

These points have been plotted on the time-series graph on sheet 1A, and each point was joined to its successor with a line segment.

3. On sheet 1A, plot the time series graph for the orbits you calculated in number 1. Examine the graph. What patterns do you observe? How do they relate to what you observed in number 1?

4. Plot the time-series graph for the orbits you calculated in number 2. Examine the graph. What patterns do you observe? How do they relate to what you observed in number 2?

5. For each equation, find the point whose connected time-series graph would be a horizontal line. Explain how you found it. Do the time-series graphs for other seeds intersect this one?

Dynamical systems can be used to model how your body deals with such medications as cold remedies. Once you ingest some medication, your body eliminates it from your system in such a way that a fixed percent of the amount remaining in your system is removed in each time period.

You will study how your body eliminates an imaginary medication called FluRidder. Assume that—

- your body eliminates 32 percent of the FluRidder that is in your system per hour.
- you have taken 100 units of FluRidder initially.
- you take an additional hourly dose of 40 units beginning one hour after you took the initial dose.

1. *(a)* What percent of the FluRidder is *not* eliminated at the end of an hour?
   (b) If you had x units of FluRidder at the beginning of an hour, how much would you have at the end?

2. Explain why you can iterate the function $y = x - 0.32x + 40$, or $y = 0.68x + 40$, to model the amount of FluRidder in your system. What does the x stand for? What does the $0.68x$ stand for? What does the 40 stand for? What does the y stand for?

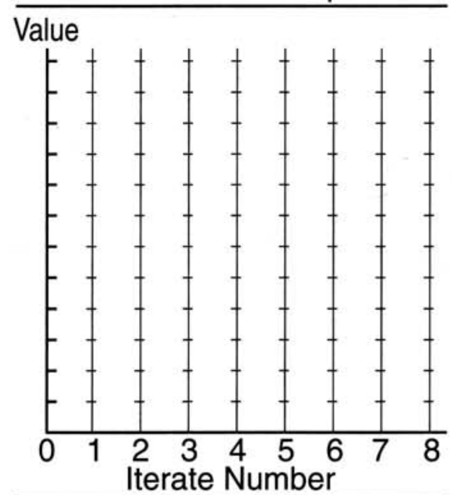
3. Make a time-series table and graph using initial doses of 40, 80, 120, 160, and three of your own choosing, to show the amount of medication in your system over an 8-hour period.

Time-Series Table

Seed	Iterate Number								
	0	1	2	3	4	5	6	7	8
40									
80									
120									
160									

$y = 0.68x + 40$

Time-Series Graph



4. Suppose that you need at least 80 units of FluRidder in your system for it to be effective. If you take an initial dose of 20 units instead of 80 units, how long must you wait before feeling the effects of FluRidder?

5. Suppose that you are an Olympic athlete and you have been taking FluRidder to help you recover from a cold. You have been taking it for three days, starting each day with an initial dose of 100 units at 6:00 A.M., and taking hourly doses of 40 units until 10:00 P.M. You learn at 1:30 P.M. on Monday that FluRidder is on the list of banned substances and that you must take a test that will detect 0.01 unit of the drug. If you stop taking the medication, when is the earliest time you could take the test and pass?

From the *Mathematics Teacher,* February 1997

Dynamical systems can also be used to model how financial institutions maintain different types of customer accounts. In what follows, assume that—

- the bank pays you 1 percent interest per month.
- at the end of the first month, you begin making regular monthly deposits of \$75.

1. Explain why you can iterate the function $y = x + 0.01x + 75 = 1.01x + 75$ to model the growth of your balance in the account. What does the x stand for? What does the $0.01x$ stand for? What does the 75 stand for? What does the y stand for?

2. Suppose that you were saving to purchase a racing bicycle, which costs \$985. How long would it take you to have that much money in your account if you started with \$100 in the account and made your first deposit at the end of one month?

3. Suppose that instead of waiting for your savings to grow, you borrowed the \$885 you needed, on top of the \$100 you had, using the same type of account, that is, 1 percent monthly interest and monthly payments of \$75. Explain why you can think of this situation as having a negative balance in the account. You have to keep making monthly payments of \$75 until your balance is \$0. How long would it take you to pay off the loan? How much would the bike actually cost you?

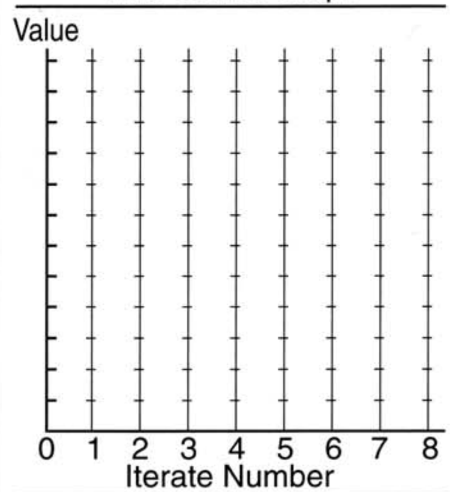
4. Make a times-series table and graph for $y = 1.01x + 75$.

Time-Series Table

Seed	Iterate Number								
	0	1	2	3	4	5	6	7	8

$y = 1.01x + 75$

Time-Series Graph



Many states have what are called Megabucks lotteries. The payoffs in these lotteries are usually \$1 000 000, but the winner does not get the money right away. Most states pay the winner by paying \$50 000 right away and then making \$50 000 payments for the next nineteen years. To finance the payments, the state places an initial deposit in an account that pays interest, makes annual payments of \$50 000, and has a final balance of \$0 after the nineteenth payment is made.

5. *(a)* Suppose that the state uses an account that pays 10 percent annual interest, that is, interest is paid once a year at a rate of 10 percent per year. Explain why the function $y = 1.1x - 50000$ models this situation.

   (b) Find the initial deposit.

To work on this sheet, you will need to refer to your time-series graphs from the previous sheets.

1. What should be the initial dose of FluRidder if you want the amount in your system to be back to that same level after you take your first hourly dose one hour later? Your body eliminates 32 percent of the FluRidder in your system, and you take a 40-unit-hourly dose. See sheet 2.

2. Someone with shaky mathematics understanding takes out a loan at 1 percent monthly interest and makes monthly payments of $75. See sheet 3. However, the balance due remains constant, month after month. What was the amount of the loan?

These situations demonstrate examples of *fixed points.* A fixed point for a function is the value of x for which $y = x$. For example, in the birthday problem (see sheet 1A), if you start with a seed of 8 and iterate the function $y = 0.5x + 4$, all the iterates equal 8.

- To check if a number is a fixed point for a function, substitute it for x and see if you get an equal value for y.
- To find the fixed point for $y = mx + b$, solve the equation $x = mx + b$.

3. What does the orbit of a fixed point look like on a time-series graph?

4. Find the fixed points for these functions.
   (a) $y = 0.9x + 10$ *(b)* $y = 2.8x - 12.4$

5. *(a)* Use algebra to find a formula for the fixed point of the function $y = mx + b$ by solving the equation $x = mx + b$ for x.
   (b) Use this formula to check the fixed points you found in numbers 1 and 2.

As you probably noticed, all orbits in the birthday problem get closer and closer to the fixed point, 8. Similarly, all orbits in the FluRidder problem get closer and closer to the fixed point, 125. The same thing did not happen with the financial problem, even though a fixed point was given.

6. What happens to the orbits of seeds that are close to 125 when you iterate $y = (0.68)x + 40$? Be sure to try both larger and smaller values. Explain why 125 is called an *attracting fixed point.*

7. What happens to the orbits of seeds that are close to –7500 when you iterate $y = 1.01x + 75$? Be sure to try both larger and smaller values. Explain why –7500 is called a *repelling fixed point.*

To understand this situation, look at the time-series graphs you made on sheets 1–3. In some graphs, the lines move closer to one another and to the fixed point's horizontal line as you move from left to right. In other graphs, the lines move away from one another and from the fixed point's horizontal line.

8. What feature of the equation seems to determine whether the lines move toward one another or away from one another? If necessary, make additional time-series graphs to check your conjecture.

9. These questions are about the equations in number 4:
   (a) Predict which function would yield a time-series graph where the lines move away from one another and which would yield a graph where the lines move toward one another.
   (b) Predict which one has an attracting fixed point and which, a repelling fixed point.
   (c) Check your predictions.

10. Write a summary of everything you know about iterating linear functions and their applications to real-life problems. Be sure to mention fixed points and the role of the parameter m in $y = mx + b$ to discuss examples you made up, and to illustrate them with time-series tables and graphs. Your summary will be evaluated on the basis of—
    - meaningful and mathematically correct examples,
    - correct use of terminology,
    - correct use of time series table and graph, and
    - clarity and neatness.