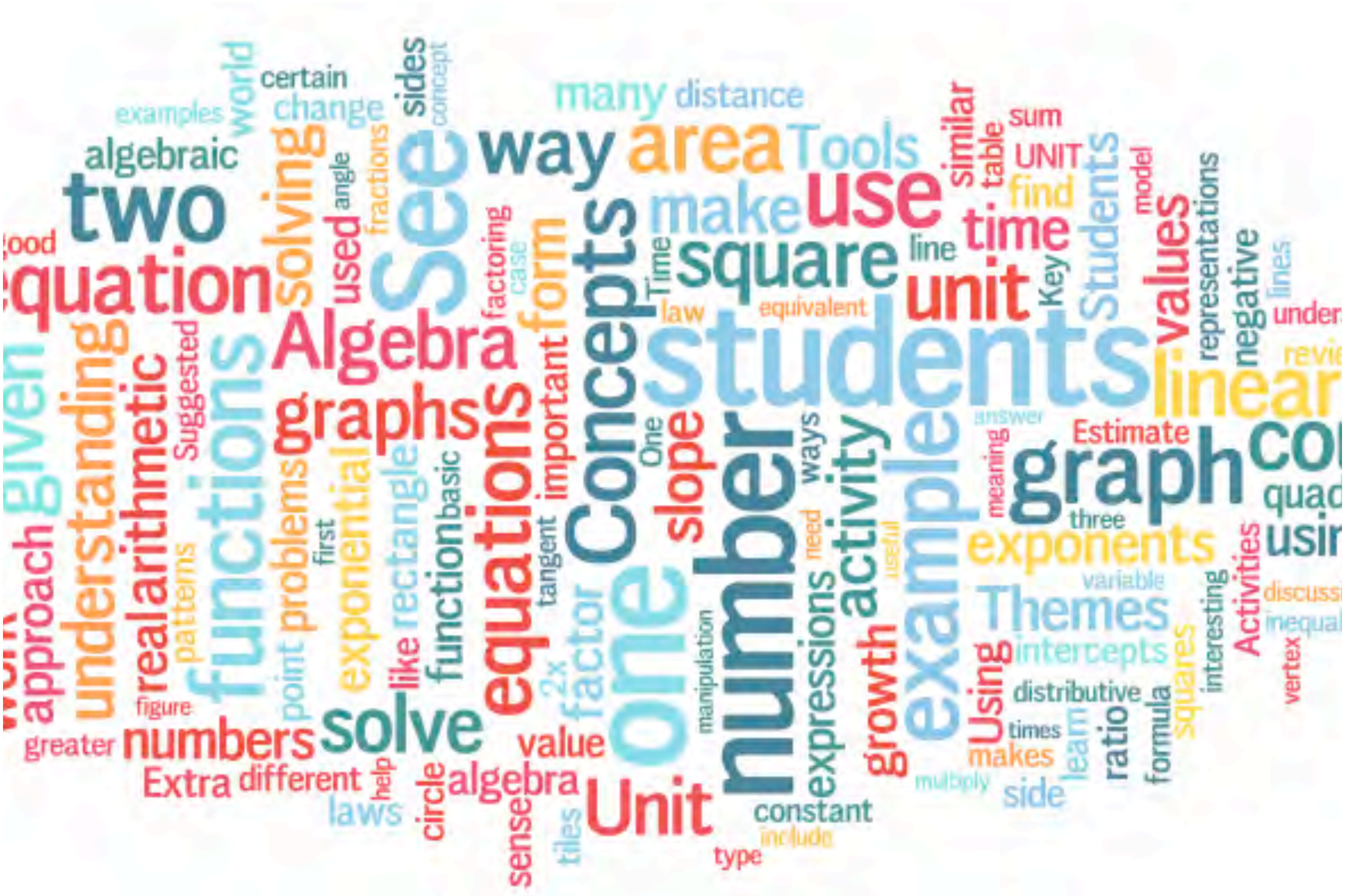

MATHEMATICS OVERVIEW

A COURSE OUTLINE

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ACKNOWLEDGMENTS

I would like to extend my thanks to Ann Shannon, who funded the development of this outline.

Amanda Cangelosi (University of Utah, Salt Lake City) and Allison Krasnow (Willard Middle School, Berkeley, CA) provided valuable suggestions.

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Introduction

This is a proposed outline for a one-year course that covers the basics of secondary school math. The course prioritizes algebra, but includes a certain amount of arithmetic, geometry, and trigonometry. The outline is based on the following assumptions:

- The goal of the course is to improve the students' relationship to math.
- The key ingredients in doing that are reasoning and sense making.
- Skills cannot be separated from understanding.
- The availability of new electronic tools cannot be ignored when prioritizing topics.

I will elaborate on these ideas in this introduction.

Technology and its Implications

Any student with access to a browser can compute, solve, factor, graph, simplify, distribute, and so on — merely by going to www.WolframAlpha.com, typing some characters, and pressing "return".

This does not mean that there is nothing to learn about all this. Quite the opposite: a student who cannot compute, solve, etc. will not know how or when to use this powerful technology. But it does mean that *it is no longer appropriate to prioritize accuracy and speed* in doing those things. No student will be able to compete with this Web site, nor should they need to.

We are now in an era where we can no longer hide behind a "skills first, understanding later" paradigm, and use it to shut the door in the face of some students. Skills and understanding cannot be separated. A student who can explain a technique, but who might make mistakes in carrying it out, is in fact better prepared than a student who has memorized the technique, can execute it correctly on the quiz, but will soon forget it because of lack of understanding.

Memorization is often a good thing, but it needs to be grounded in understanding, not be a substitute for it. If a student has memorized something without understanding, it is the teacher's responsibility to complement and scaffold those rote skills rather than pretend the topic is mastered.

Another consequence of this state of affairs is that while some drill can be helpful, it is best to get practice in non-random, interesting drills than in random, boring drills that will kill the student's motivation. It is also useful to mix it up: sets of exercises can focus on one topic, but they can also make connections and include review of previously studied topics.

Non-Cognitive Skills and Attitudes

"Non-cognitive" skills and attitudes are at least as important as the mathematical skills that students will need. They include:

• Communication skills: students need to be able to talk and write about math.

• Collaboration skills: students need to be able to work with each other.

• Resilience and perseverance: students need to "stick with it" across challenges.

• Self-confidence and self-advocacy: students need to trust themselves and know when and how to seek help when needed.

These skills are of course essential throughout the course, and in combination they matter a lot more than any particular math skill when it comes to success in school, in life, and yes, in mathematics.

Pedagogy

For most of the students who take it, much of this course is review of previously seen material, and it risks to generate an "Oh, no! Not this!" reaction. It is essential to success that this course not feel like the courses which the students have previously taken. Teaching the same material in the same way will not lead to different outcomes.

Algebra and symbol sense are our main destination, but we can get there more effectively by using multiple representations of the key concepts. Whenever possible, I propose manipulative, graphical, geometric, and/or numerical approaches to complement algebraic manipulation. This is both to provide different access points to different sorts of learners, and to provide deeper understanding to all. Expertise is largely the ability to transfer knowledge between different representations.

Tools

Another key pedagogical concept is the use of learning tools — manipulative and electronic. This too is intended to help reach a wide range of learners, and to deepen understanding for all. It is not intended as an alternative to traditional paper-pencil techniques. Rather, it is intended to help students master those.

The manipulative tools are algebra tiles or blocks, and geoboards or dot paper. The electronic tools are graphers, interactive geometry software, computer algebra systems, and specialized applets.

Algebra Tiles or Blocks

Algebra tiles or blocks help anchor several key ideas, by providing a geometric model of variables and operations. We use them especially to introduce the distributive rule and factoring, and to illustrate key ideas about quadratics, including completing the square. See Units 1, 2, 9, and 13.

There are multiple versions of these manipulatives. The activities referenced in this outline are based on the Lab Gear, but could be carried out with any of them. For a comparison and history of the various algebra manipulatives, see http://www.mathedpage.org/manipulatives/alg-manip.html.

There is an implementation of algebra tiles on the Web at the National Library of Virtual Manipulatives, from Utah State University: http://nlvm.usu.edu/en/nav/frames_asid_189_g_1_t_2.html.

Geoboards / Dot Paper

The geoboard is a useful environment to discuss slope, and especially area, distance, and the Pythagorean theorem. If geoboards are unavailable, one can use dot paper for those activities. See Units 4 and 8.

The University of Cambridge offers an online geoboard: http://nrich.maths.org/2883.

The Ten-Centimeter Circle

The ten-centimeter circle is a great environment to get started with trigonometry. It is available from http://www.mathedpage.org/circle/.

Calculators

While the outline includes much number-focused work, the emphasis is on understanding, not speed or accuracy in calculation. Thus, except perhaps in specific lessons, there is no reason to forbid the use of the calculator. In an age where calculations are carried out electronically in every single institution, it does not make sense to disallow this in the classroom.

Electronic Graphers

Electronic graphing is now standard in the secondary math classroom. The choices are many, from graphing calculators to various implementations on computers, tablets, smartphones, and the Web. We use electronic graphing in Units 3, 6, 11, 13, and 14. It may also be useful elsewhere.

Multipurpose Electronic Tools

Interactive geometry allows students to construct and manipulate geometric figures on a computer screen. It is particularly useful in Units 4 and 15.

A *calculator / computer algebra system* (CAS) will automatically solve equations, factor polynomials, and so on. I recommend CAS in Units 5 and 11, but it may also be useful elsewhere.

An open source application for interactive geometry, graphing, and CAS is GeoGebra, available from http://www.geogebra.org/. Another is available from the National Council of Teachers of Math (Go to http://www.nctm.org/resources/high.aspx and click on “Core Math Tools” in the sidebar.)

On the Web, http://www.wolframalpha.com offers CAS and graphing capabilities.

Of course, these tools work best if students have access to computers, but some of the activities can be done by the whole class if the teacher can project a computer screen.

Online Applets

I reference a number of single purpose online applications, known as "applets". The ones from the Concord Consortium's "Seeing Math" page: http://seeingmath.concord.org/sms_interactives.html are used in Units 3, 11, and 13. Note that each of these applets is accompanied by worthwhile sample activities.

How to Use this Outline

This outline offers a structure for a course that covers most of the basics of secondary school math. The content of each unit is presented with a list of key concepts, followed by a list of suggested activities. There is no attempt to break up the unit into individual lessons. Instead, I hope that curriculum designers and teachers who use the outline will take both the concepts and the activities into consideration when organizing the unit into lessons. They will also need to decide on what homework, drill, exposition, and assessments are required to complete the unit.

Most units are intended to take about two weeks to complete. In some cases, depending on how they are implemented, it may mean that the suggested content may not all fit in the allotted time period. In fact, the second half of the course includes a number of concepts and activities that are labeled "Extra". Those tend to be more challenging, and can be omitted without harming the overall sequence.

The proposed activities are mere suggestions, presented in order to articulate the mathematics that the unit is intended to cover. The links to specific curricular materials are intended to show one way to get there with an approach that is based on reasoning and sense making. Certainly there are other ways to get to the same ideas, but having a model is useful, especially in the units where the approach is decidedly non-traditional.

The resources referenced the most often are the lessons from *Algebra: Themes, Tools, Concepts* (www.MathEducationPage.org/attc) by Anita Wah and Henri Picciotto, the labs from *Geometry Labs* (http://www.MathEducationPage.org/geometry-labs), and the activities from *The Algebra Lab: High School* (http://www.MathEducationPage.org/manipulatives/lab-gear.html), both by Henri Picciotto. All three include notes for the teacher. In combination with the notes in this outline, those may be useful to the end user of this curriculum. The thinking behind these materials, and behind this outline, is spelled out in more detail there, as well as in various articles on my Math Education Page.

Sequencing

The topics are sequenced in a non-traditional way. Most students in this class will have already seen most of these topics before. This makes it less important to stick to a supposedly logical sequence of topics, or to an arbitrary traditional sequence.

Opening the course with arithmetic review, for example, would be sure to be discouraging to some students who have found this overwhelming in the past. At the same time, it is useless to others, who have already mastered those basics and find the review boring, if not insulting. Of course these topics are important, but by design this course will approach them at various times across the year, not right up front.

Because trouble with algebra is one of the main obstacles to further work in math and science, all the odd-numbered units focus on algebra. Units 2, 6, 10, and 14 focus on numbers, usually from a high school point of view, by making connections with graphing, geometry, functions, and algebra. Units 4, 8, and 12 are largely about geometry, but they make connections with arithmetic and algebra.

Important topics surface repeatedly, and related topics are not necessarily treated consecutively. For example, we work on the distributive law and factoring in Units 2, 9, and 13. This is very helpful to students, because it gives them opportunities for review, it extends the exposure to key ideas, and it gets the message across that we are not "done" with the distributive law at the end of a certain chapter. However, not keeping all the lessons on a given topic together is challenging for teachers, which is why I supply a listing of where the key recurring concepts appear. (See below.)

Finally, both within each unit, and in the overall design of the course, I tried to delay complicated symbol manipulation, preferring to ground that sort of work in numerical, graphical, and geometric representations.

Topic Selection

Since I was trying to cover the highlights of secondary school math in a single year, I had to decide which topics would be included, and (more difficult!) which would be omitted. I used a combination of criteria in making my selection. What topics do community college teachers want their students to already know coming in? What topics are foundational to further work in math and science? What topics can be made both interesting and accessible?

Other choices would have been possible, but I ended up omitting the following:

- Solving inequalities by algebra
- Rational expressions
- Arithmetic and geometric series
- Rational exponents other than ½
- Anything beyond a basic introduction to logarithms and trigonometry
- Probability and statistics

… plus no doubt other topics I am not thinking of right now. Still, a student who masters the material that is included in this outline will be in a strong position to move to the next level and learn those things.

The Main Strands

Here is a list of the topics that surface repeatedly.

Area

Units 1, 2, 4, 6, 8, 13, 15

Arithmetic and Number Sense

Units 2, 6, 10, 14

Distributing and Factoring

Units 2, 9, 10, 13, 14

Equations and Inequalities

Units 2, 3, 5, 9, 11, 13, 14

Functions

Units 1, 3, 6, 7, 11, 13, 14, 15

Graphs

Units 1, 3, 4, 5, 6, 7, 9, 11, 12, 13, 14

Powers and Roots

Units 3, 7, 8, 10, 14, 15

Quadratics

Units 3, 7, 9, 13, 15

Trigonometry

Units 7, 12

MATHEMATICS OVERVIEW COURSE OUTLINE

Unit	Key Concepts	# of Weeks
1 Puzzles and Patterns	Variables; Functions; Multiple Representations; Perimeter and Area	2
2 Numbers and Operations	The Distributive Law; Factoring; The Meanings of Minus; Parentheses and Order of Operations; Equations and Identities	3
3 Understanding Graphs	Graphs as Models; Graphs and Tables; Intercepts; Slope; Special Graphs	2
4 Scaling and Proportion	Similarity, Scaling, and Dilation; Equivalent Fractions; Equivalent Fractions and Slope; Ratio of Areas	2
5 Equations and Inequalities	What Does It Mean to Solve an Equation?; Why Does One Solve Equations?; There Are Many Ways to Solve Equations; Algebraic Techniques; Solving Inequalities	3
6 Numbers and Graphs	Signed Number Arithmetic; Fraction Arithmetic; Introduction to Function Diagrams; Multiple Representations	3
7 Growth and Change	Rate of Change; Constant Rate of Change, Exponential Growth and Decay; Arithmetic vs. Geometric Sequences; The Tangent Ratio	3
8 Area and Distance	Squares and Square Roots; Taxicab vs. Euclidean Distance; The Pythagorean Theorem	2
9 Trinomials and Identities	Distributing and Factoring; Factoring Strategies; The Zero Product Principle; Multiple Representations of Trinomials; Factored Form of Quadratic Functions	2
10 Powers and Roots	The Laws of Exponents; Zero and Negative Exponents; The Exponent ½; Extra: Working with Radicals	2
11 Satisfying Constraints	Solving Systems of Equations: What Does It Mean?; Multiple Algebraic Forms for Linear Functions; Solving Linear Systems: Different Strategies; Extra: Graphing Inequalities in Two Variables; Extra: Linear Programming	2
12 Angles and Ratios	Solving Right Triangles; The Sine and Cosine; The Basic Trig Functions	2
13 Parabolas and Quadratics	Quadratic Forms; Extra: Algebraic Connections; Equal Squares and Completing the Square; The Quadratic Formula	2
14 Using Exponents	Real Number Exponents; Compound Interest; Scientific Notation; Extra: Square Roots and Absolute Value; Extra: Logarithms	2
15 Functions and Modeling	N^{th} Power Variation; π; Iterating Functions	2
	TOTAL:	34

Unit 1: Puzzles and Patterns

Overview

The purpose of this unit is in some ways the purpose of the whole course: trying to get across the idea that math is both accessible and interesting. We focus on foundational concepts in algebra: variables and functions in their multiple representations. In an approach that is typical of all the units in this course, we start with concrete examples, so everyone can engage, but we end at an appropriately abstract level.

At least as important as the mathematical skills outlined below are "non-cognitive" skills and attitudes, such as the ones mentioned in the introduction. This first unit is crucial in setting the tone for the rest of the course.

Time Estimate

2 weeks.

Key Concepts

Variables

Variables, of course, are a key to doing any work in mathematics and science. Merely "explaining" variables to students does not usually work. It is preferable to give them frequent experiences generalizing from the concrete to the abstract. The concept is so difficult for some students that it is foolish to make a full grasp of it a necessary prerequisite for further work. On the other hand, it is so pervasive that it is possible to take weeks or months to develop the needed understandings — as long as most of the lessons start in the concrete realm.

Functions

We save many concepts about functions for later in the course, but the first unit is a good time to introduce the essence of the concept: *a function assigns a single output to each input*.

Functions are important not only because of their central importance in all of mathematics and science, but also because they can be used to help teach ideas about numbers, operations, and modeling (as we will see later, mostly in Units 6, 7, and 15.)

In this introductory unit, we will use visual contexts to make the point that functions are one way to describe things in the real world. Those contexts are chosen because while they are undeniably real, they are simpler than the ones offered by the real "real world".

Multiple Representations

This concept is actually several concepts in one: the functions we will prioritize in this course can be represented in multiple ways:

- in words;
- as a table of values;
- as a graph;
- as a formula.

Again, this is an understanding that will deepen over time. For weaker students, some of the representations may offer an entry point that a strictly symbolic approach does not offer. Over time, navigating between the representations helps them develop their understanding of other representations. For stronger students, being able to navigate between the representations helps them develop a deeper understanding. For all students, the availability of multiple representations provides a vehicle for communication, a handle on how to think and talk about functions. (We will use a fifth representation, the function diagram, in Units 6, 7, and 15.)

Perimeter and Area

These concepts are of course basic geometric ideas. They are featured in this unit to provide an accessible and interesting context to learn about the multiple representations of functions.

Some students may have memorized formulas for area and perimeter, without really understanding what the formulas were about. They might say things like "area is when you multiply", or "area is length times width", which of course is sometimes, but not always true. The emphasis here is much more basic, and in many cases area and perimeter will be found by just counting.

Confusing area with perimeter is common. Giving students time to explore these topics simultaneously allows them to create a more concrete idea of what each measurement represents. This of course does not prevent students from discovering and using formulas at some future time (see Unit 8) — in fact, it makes such future learning more likely to succeed.

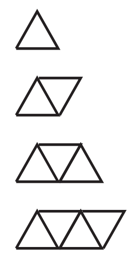
Suggested Activities

Perimeter and Area on Graph Paper

In this activity, students make simple closed shapes following the lines on graph paper, and investigate the relationship between the area and perimeter of the shapes. The basic questions are: "For a given area, what is the maximum perimeter? What is the minimum?" The first question is not too difficult; the second is quite challenging — in a good way. The only prerequisite is the ability to count. The activity would work well as an anchor to the unit and the whole course, as everyone can participate, and the exploration is rich, leading to a table of values, to a linear and non-linear graph, and to connections between visual and numerical patterns. See *Geometry Labs 8.1-8.3*.

Visual Patterns

In this activity, students study visual patterns such as the one on the right. They answer questions like: What is the perimeter of the tenth figure in the sequence? the 100th one? The nth one? They can explore other questions about the same visual patterns, such as how many "toothpicks" is it made of? Dozens of 2D and 3D patterns of this type can be found on the site http://www.visualpatterns.org/. [The site has some technical issues, which hopefully will be fixed by the time you read this!] And moreover, students can create their own patterns, leading to further explorations.

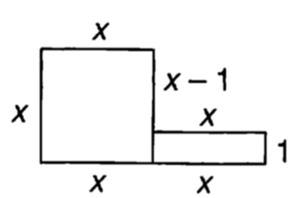


What's My Function?

This is a game, which can be supplemented by any number of worksheets. The teacher thinks of a function, such as $y=3x+2$. Students ask questions like "If x is 7, what is y?" and the teacher answers with a number. The functions can be calibrated to the level of the class. Easy and difficult functions can alternate. When a student knows the function, instead of giving away the answer, he or she can play the role of the teacher.

Introduction to Algebra Tiles

The first unit is a good time to introduce algebra tiles or blocks, which will make several appearances in this course (in Units 2, 9, and 13). The activity should include the basics, especially the area model, which is the underlying concept of why, for example, x^2 is a square. In addition to the basics, students can be asked to find the perimeters of algebra tile figures. Comparing answers provides a strong motivation for combining like terms. For example, the perimeter of this figure is $6x$:



Such perimeter problems can be found in *The Algebra Lab: High School* (see p. x for a list, including challenges where the students are asked to arrange certain blocks in order to obtain a given perimeter.)

What's in the Bag?

Another introductory activity with algebra tiles is "What's in the Bag?" (or "What's in the Box?"). This is a variation on "What's My Function?" The teacher puts a few blocks in a bag, and students ask questions like "If x is 7, what's in the bag?" If the tiles include y-tiles as well as x, the game is of course more challenging, as the question becomes something like "If x is 7 and y is 1, what's in the bag?"

Unit 2: Numbers and Operations

Overview

This unit is not an encyclopedic review of arithmetic. It focuses on just two important ideas: the distributive law, and minus (its meanings and manipulation.) Those are foundational in algebra, and many students are severely hampered because they do not understand them. The unit also includes lessons on estimation and mental arithmetic, some of which are based on the distributive law. (More arithmetic is coming in Units 4 and 6.)

Time Estimate

3 weeks.

Key Concepts

The Distributive Law

The distributive law ties together multiplication and addition, which makes it an essential concept in both arithmetic and algebra. Many students are familiar with "FOIL" as an approach for multiplying binomials, but this usually does not come with an understanding of the underlying structure. As a result, students who only know this often forget it, or have trouble dealing with situations other than "binomial times binomial." It is much more effective to base this work on a geometric model: the area of a rectangle. This is a robust approach, which generalizes to multiplying any combination of polynomials, and also throws light on related issues in arithmetic.

Factoring

Some calculators and Web sites will factor any factorable number or polynomial instantly. Thus, factoring, as a skill, is not as important as it used to be. However, it remains essential on a conceptual level: a student does not fully understand the distributive law if he or she cannot factor anything. Fortunately, the algebra tiles model turns factoring into a visual puzzle, and puts the concept within reach of all students. While the unit prioritizes factoring algebraic expressions, it is helpful to include a bit of work on factoring whole numbers, as that will be useful in Unit 6 when working on the arithmetic of fractions.

The Meanings of Minus

Before the manipulation of minus becomes automatic, it is necessary to understand the different meanings of that symbol depending on context. In front of a number, it means "negative". In front of any expression, it means "the opposite", and between expressions, it denotes subtraction. Students who have struggled in math often don't believe that subtraction is equivalent to adding the opposite. Giving them time to make sense of this notion is important.

Parentheses and Order of Operations

Likewise, students need to learn how to read somewhat complicated expressions and understand what they say. This is the understanding that underlies correct use of parentheses and order of operation.

Equations and Identities

All the above concepts come to life in the discussion of equations vs. identities. An equation presents two expressions as equal. If they are equal for any value of the variable, we have an identity. If they are equal for specific values only, those values are solutions to the equation. It is also possible that the two sides are never equal and the equality is always false. Understanding these differences is an important prerequisite to equation solving and to algebraic literacy in general.

Suggested Activities

Make a Rectangle

Students are given some algebra tiles, and need to arrange them into a rectangle. Since the area of a rectangle = length times width, what students have done, in effect, is factored a polynomial. This is an activity that is essentially without prerequisites, which is why it is good to start with it. It sets the stage for a discussion of the distributive law and factoring. For example, two x^2 tiles and four x tiles can be arranged in an $(x+2)$ by $2x$ rectangle:

$2x^2 + 4x = 2x(x + 2)$

"Make a Rectangle" puzzles can be found in *The Algebra Lab: High School* (see p. ix for a list.)

This activity and the next one should be done without any minus signs. While it is possible to extend the area model of multiplication to include negative numbers, it is somewhat tricky, and not worth doing in this course. Once students master this without minus, generalizing is straightforward. The problems in this unit should be accessible. We'll return with more challenging examples in Unit 9.

Related Algebra Tile Activities

A visual version of the distributive law: given the two sides of a rectangle, e.g. 2x and (x+2), what is the area? This can be answered by making the rectangle in question with tiles.

A visual representation of division by a monomial: given a rectangle and one side of it, what is the other side? For example, if you make a rectangle using two x^2 tiles and four x tiles, with one side of length 2x, what's the other side? That question represents $\frac{2x^2+4x}{2x}$.

Examples of both types of problems can be found in *The Algebra Lab: High School,* Chapter 5.

The Box

Of course, the point of the above activities is not for the students to become algebra tile experts. It is to transition to the symbolic version of these activities. An excellent way to do it is "the box", also known as the generic rectangle. Here is an example:

	x	$+2$
$2x$	?	?

Distribute

	?	?
?	$2x^2$	$4x$

Factor

	?	?
$2x$	$2x^2$	$4x$

Divide

This approach generalizes to any size polynomial, and once multiplying with minus is understood, that too can be done in the box.

Number Talks

Besides its usefulness in every day life, mental arithmetic is a good arena to apply the distributive law. For example, to multiply 7 by 58, one can multiply 7 by 50, and add the result to 7 times 8. Or, multiply 7 by 60, and subtract 7 times 2. Discussions of such problems, called "Number Talks", are a good way to build student self-confidence and an understanding that there are many ways to get to "the answer". Students can be asked to find as many of those ways as possible. It is also a good context to discuss approaches to estimation. (For example, 7 times 58 cannot be more than 7 times 60.) See http://www.insidemathematics.org/index.php/classroom-video-visits/number-talks.

Reading Expressions

One ingredient, perhaps the main one, in effective symbol manipulation is the ability to read algebraic expressions. For example, is $2x(x+2)$ a multiplication or an addition? Can you write an equivalent expression that is an addition? This example is of course related to the distributive law, but similar discussions of expressions are also possible to clarify the meanings of minus and the use of parentheses.

Always, Sometimes, or Never?

Is it always, sometimes, or never true that $-2(x + 3) = -2x + 3$? What about $-2(x + 3) = -2x - 6$? What about $-2(x + 3) = x + 6$? Problems of this type can be created for all the key basic concepts of algebra. They lend themselves to discussion, and help students make sense of the symbols. Students can substitute various values for the variable to test their conjectures. See http://www.mathedpage.org/attc/eq-vs-id.pdf. We will return to this question, using technology, in Unit 9.

Factoring Numbers

After reviewing how to recognize divisibility by 2, 3, and 5, students learn to factor whole numbers, using the factor tree format. This can be accompanied by a discussion of prime factors, common factors, and common multiples.

UNIT 3: UNDERSTANDING GRAPHS

Overview

A deep understanding of graphs is multifaceted. In this unit, we zero in on how graphs can model the real world, and on a basic technical understanding of graphs. At this level, the latter includes the connection between graph and table of values, key ingredients such as slope and intercepts, and some fundamental cases: horizontal lines, vertical lines, lines through the origin, and recognizing the essential characteristic of linear vs. nonlinear graphs. We do not emphasize graphing functions point by point on paper, preferring to get at the same ideas in other ways, and to seek more meaning and connections. We will do much more with graphs in future units, starting with Units 5–7.

Time Estimate

2 weeks.

Key Concepts

Graphs as Models

Graphs can represent models of the real world. A very standard and important example is time–distance graphs, with time on the x-axis, and distance traveled on the y-axis. Examples with time on the x-axis are plentiful (e.g. temperature as a function of time of day.) More generally, it is traditional to have the input variable on the x-axis, and the output variable on the y-axis. (Traditional terminology for this is independent and dependent variable.)

Graphs and Tables

One way to visualize graphs as models is by making a table of values for a given situation or formula, and creating a graph point by point. If it makes sense to do so (in the case of a continuously varying variable) one can connect the dots. Conversely, one can create a table of values from a graph.

Intercepts

The intersections of a given graph with the x- and y-axes are called the intercepts. Intercepts don't always exist. If there are values of x for which $y = 0$, those are the x-intercepts. If there is a value of y when x=0, it is the y-intercept. If the y-intercept and an x-intercept are both 0, the graph passes through the origin. If a graph models a real-world situation, the intercepts have a specific meaning for that situation.

Slope

Most students taking this class will have heard of slope as "rise over run". This is not wrong, of course, but it is a limited understanding. At this point, introduce slope triangles as a way to visualize rise and run. Be cautious in discussion of "steepness", as actual steepness on a graph depends in part on what units and scales are used on the *x*- and *y*-axes. A key idea is that if slope is unchanging, then the graph is a straight line. If it changes, then it is a curve of some sort. We'll return to slope and rate of change, especially in Units 4 and 7.

Special Graphs

If students do not understand horizontal and vertical lines, they are probably not clear on many of the other concepts in this unit. It is important to use those special cases as ways to throw some light on how graphing works. Other linear graphs and the slope-intercept form of their equations are of course worth knowing about, as they are so basic to all further work in algebra. This is also a good time to get acquainted with the appearance of basic nonlinear graphs, particularly those corresponding to quadratic and exponential functions.

Suggested Activities

Qualitative Graphs

These are activities based on reading graphs that do not actually involve a numerical scale. They allow the discussion to focus on descriptions like "the quantity is increasing", "the truck is stopped at a traffic light", "the liquid is pouring out faster and faster". A free online applet from the Concord Consortium's Seeing Math set, the Qualitative Grapher, allows you and your students to easily make up and discuss qualitative graph stories. (Go to http://seeingmath.concord.org/sms_interactives.html.) Include discussion of intercepts in those scenarios.

Interpreting Graphs

Graphs as "real-world" models can generate interesting discussions, as well as seed many important algebraic concepts. In particular, the meanings of slope and intercepts in each model deserve attention. Numerous examples can be found, for example, in *Algebra: Themes, Tools, Concepts* (www.MathEducationPage.org/attc/attc.html), Chapter 3, Lesson 8, and Chapter 4, Lessons 1, 6, 10. "The Bicycle Trip" (Chapter 4, Lesson A) is a particularly entertaining example.

On or Off the Graph? Above or Below?

Given an equation, predict whether a particular point is on its graph, or above it, or below it. Students can easily do this by substituting the *x* value for the given point, and calculating the resulting *y* value, but this may not have been emphasized in previous courses. This activity requires and supports a foundational understanding of the relationship between point, equation, and graph. That relationship is sometimes called the "Cartesian connection". Do not limit this activity to linear functions.

Which Is Greater?

Given two expressions that each involves a variable, which expression is greater? For example:

$x + 3$ is greater than $x - 2$.

$3x$ is greater than $2x$ if x is positive.

$3x$ and $2x$ are equal if x is 0.

$2x$ is greater than $3x$ if x is negative.

More complicated examples can readily be sorted out by inspecting the graphs. For those, use an electronic grapher. For example, inspecting the graph of $y = 2x + 1$ and $y = 3x$ shows that

the expressions are equal if $x = 1$;

$2x + 1$ is greater than $3x$ if $x < 1$;

$3x$ is greater than $2x + 1$ if $x > 1$.

Do not expect students to spontaneously think of solving an equation to sort this out at this stage. Certainly if they do, that is fine, but the problem is not fully answered just by finding that $3x = 2x + 1$ when $x = 1$. To fully answer, it is necessary to have a strategy to figure out what happens when x is not equal to 1.

$y = mx + b$

The slope-intercept form for linear expressions can be explored using the Concord Consortium's Seeing Math Function Analyzer (http://seeingmath.concord.org/sms_interactives.html).

Make These Designs

After that review of slope-intercept form, it is time to reverse the traditional "here's an equation, make the graph" activity. The idea is to give students interesting designs made entirely with straight lines, and ask them to find the $y = mx + b$ formulas needed to create those designs on an electronic grapher. There is an example, accompanied by detailed teacher notes, at http://www.mathedpage.org/calculator/make-these/. We will have a more advanced "Make These Designs" activity in Unit 13.

UNIT 4: SCALING AND PROPORTION

Overview

This unit focuses on similar figures as a springboard for work on proportion and fractions. For maximum payoff, we prioritize similar rectangles and right triangles, as they lead to the most important applications. Proportion of course is largely about equivalent fractions, which we discuss in various representations. (We return to fraction arithmetic in Unit 6.) Finally, we analyze what happens to the area of a figure when it is scaled.

Time Estimate

2 weeks.

Key Concepts

Similarity, Scaling, and Dilation

If two shapes have exactly the same shape, they are said to be similar, irrespective of size. For example, all squares are similar, but rectangles can have different shapes. There are different ways to mathematically guarantee that shapes have exactly the same shape. In the case of polygons, it is sufficient that the angles be equal, and the sides proportional.

A geometric transformation that creates similar figures is the dilation, where the distance between input points and the center, multiplied by a given number, yields the distance between the output points and the center, in the same direction. Figures can be dilated into similar figures even if they are not polygons, so this provides a more general definition of similar figures.

Equivalent Fractions

Proportional sides imply a scaling factor: one number we can use to multiply the side lengths of one of the shapes and get the side lengths of the other. That number is the ratio of the sides, and that relationship can be written as a set of equivalent fractions, all of which are equal to the scaling factor. Equivalent fractions correspond to the same decimal, which can be obtained by dividing the numerator by the denominator. Understanding equivalent fractions is the key to much of fraction arithmetic, as it underlies the use of common denominators. Note that the concept of equivalent fractions is the overwhelmingly central idea of this whole unit. It is approached in context, but it is never far from the surface in all the activities.

Equivalent Fractions and Slope

The slope of a line can be found by the use of slope triangles. Similar slope triangles yield equal slopes. Students need to understand that slope is a number, whether expressed as a fraction or as a decimal.

Ratio of Areas

Because area is a two-dimensional quantity, when a figure is scaled, its area is multiplied by the square of the scaling factor. For example, a 2 by 3 rectangle has area 6. If its dimensions are doubled, the resulting 4 by 6 rectangle has area 24, 4 times the original area.

Suggested Activities

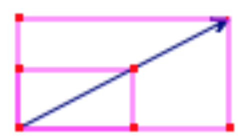
Note: A version of most activities in this unit can be carried out using interactive geometry software. In that environment, similarity is achieved through the dilation tool. Interactive geometry makes it easier to discuss similarity of shapes other than polygons, such as circles or ellipses.

Scaling on Grid Paper

Compare what happens to a grid paper figure if you stretch it horizontally by a factor of 2, stretch it vertically, or stretch it both horizontally and vertically. Only in the third case do you get a similar figure. Also compare with *adding* the same amount to the horizontal and vertical dimensions. In the case of similar figures, how do their areas compare? See *Geometry Labs*, 10.1, 10.3.

Similar Rectangles, Similar Right Triangles

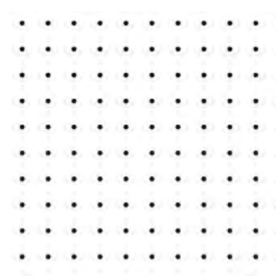
Note that the shape of a rectangle is given by the ratio of length and width. (For a 2 by 3 rectangle, that ratio is 2/3 or 3/2 depending on which side you consider to be the length.) Also note that you can scale a rectangle by extending its diagonal, as in this figure:



See *Geometry Labs*, 10.2, and *Algebra: Themes, Tools, Concepts*, 14.1.

Similar Right Triangles and Slope

Search for all the slopes of lines obtained by connecting two pegs on a geoboard, or two dots in a given array of dots, such as this 11 by 11 array:



This is in fact a review of both equivalent fractions and slope. The right triangles hanging from a line in a graph are all similar. Conversely, similar triangles yield the same slope.

See a "real-world" application in *Algebra: Themes, Tools, Concepts* 11.3.

Scale Models

Figure out the scaling factor of a given scale model, such as a printout of a Google Earth image of a school, or a toy. And, going in the other direction, figure out the possible dimensions of a scale model of real life objects. See for example a lesson along these lines geared to somewhat younger students, in *Algebra: Themes, Tools, Concepts*, 6.11, or http://mrmeyer.com/threeacts/superbear/.

Clocks, Angles, and Pie Charts

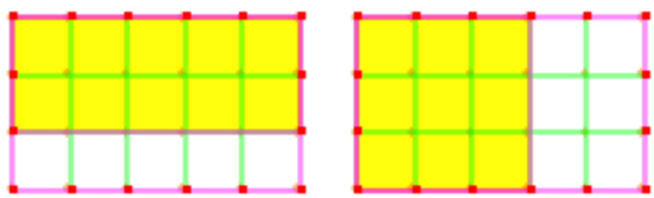
25% can be represented in a pie chart as ¼. But note that ¼ of an hour can be represented in the same way on an analog clock — and it is 15 minutes. Moreover, the angle for ¼ of the way around a circle is 90°. These equivalences offer an interesting activity, where students create figures for those various representations, and answer questions like "20 minutes is what fraction of an hour? What is it as a percent? What is the central angle that represents that fraction of 360°?" And so on, starting in one or another of these three representations.

Fractions on a Number Line or Grid Paper

You can discuss "which is greater?" about fractions using a number line model — equivalent fractions and common denominators are helpful for that! See for example:

http://www.geogebratube.org/student/m13567.

Here is another strategy, which will be useful in Unit 6: to compare 2/3 and 3/5, for example, make two 3 by 5 rectangles on graph paper. It is easy to shade in 2/3 in one, and 3/5 in the other. Then, the comparison is easy, because we actually can see that one is 10/15, and the other is 9/15:



Unit 5: Equations and Inequalities

Overview

In all likelihood, students taking this course have had the opportunity to solve linear equations before. The main purpose of this unit is to build up the conceptual apparatus needed to understand what an equation is, what equation solving is good for, and that there are multiple ways to solve equations. Some of these understandings are grounded in the number system (the use of opposites and reciprocals), and some are grounded in the idea of multiple representations of functions. We also do a little work with inequalities, based on the same principles.

Time Estimate

3 weeks.

Key Concepts

What Does It Mean to Solve an Equation?

Many students can, in their words, "get x by itself" without actually understanding what they have accomplished. One way to put some meaning in equation solving is by seeing that if you have two expressions, depending on the value of the variable, one expression may be greater than the other, or they can be equal. This was previewed in Unit 2 ("Always, Sometimes, or Never?") and in Unit 3 ("Which Is Greater?") In this unit, students put this together with the idea of equation solving, and understand that solving an equation is finding the values of x that makes the equality true.

Why Does One Solve Equations?

If a student only sees disembodied equations with no real-world context, and formulaic word problems that seem unrelated to anything outside of math class, they are likely to conclude that equation solving is strictly relevant to school math. It is motivating to see how solving equations can help answer questions students and other people have.

There Are Many Ways to Solve Equations

Many equations cannot be solved by algebraic manipulation, and require some other approach — often an approach supported by technology. Linear equations, of course, can be solved algebraically, but they too can be solved by any number of other approaches. In fact, trying to use pencil and paper only to solve an equation like $1.234567(x - 8.9098) = 7.6543x + 2.1012/3.456$ seems absurd. Other approaches include the use of trial and error, tables or spreadsheets, graphs, and computer algebra systems. Students should learn when it is appropriate to use which method.

Algebraic Techniques

The understandings that support traditional algebraic equation solving include opposites and reciprocals, undoing operations, working backwards, and especially equivalent equations. Students should be able to explain when and how these ideas are used in the process of solving a linear equation. Specifically, when they are (for example) adding the same thing to both sides of the equation, they are creating another equation, which is equivalent to the previous one, meaning that any value of the unknown that solves one will also solve the other.

Solving Inequalities

Many, but not all, of the techniques used to solve equations can be used to solve inequalities. The exception, of course, is algebraic manipulation. When solving inequalities algebraically, a whole layer of complexity is added because of the need to reverse the inequality when multiplying by a negative number. It is not necessary to include that technique in this course.

Suggested Activities

A Real-World Problem

Open the unit with a real-world problem that leads to equation solving. Not a traditional word problem, but something meatier, where two or more expressions are modeled with functions, and fully solving the problem requires the solving of equations. See for example this comparison of cell phone plans:

http://www.mathedpage.org/lessons/cell/

A lesson of this type can be the springboard for a discussion of different approaches to equation solving.

Trial and Error

One way to solve equations is by trial and error. You substitute a certain number for the variable. If the two sides of the equation are not equal (i.e., the statement is false), you try another number, trying to learn from the previous trial. One good way to organize your attempts is in an input-output table. Equations in this activity need not be restricted to linear functions.

Using a Table or Spreadsheet

If you enter the expressions on either side of the equal sign as functions in a graphing calculator, you can look at the table of values to find a value of x that makes the outputs equal. This is an opportunity to learn how to set up the tables: the first value of x, the standard change in x. In a way, this is a higher tech version of trial and error; it reveals more about the nature of the equation, and some trial and error may be needed if the exact solution happens to be between values shown on the table.

A spreadsheet can be used in essentially the same way, and of course spreadsheet skills are worthwhile well beyond the needs of this unit or this course.

Equations in this activity need not be restricted to linear functions. Also note that this approach can be used to solve inequalities.

Using a Graph

Once the expressions on either side of the equal sign are entered in a graphing utility, looking at the two graphs is a sort of condensed and visual representation of the same information that was in the table. The solution(s) to the equation, if there are any, are where the graphs intersect. It is possible to solve any equation this way, but it requires some skill in finding the appropriate window. Equations in this activity need not be restricted to linear functions.

Of course, as we saw in Unit 3 ("Which Is Greater?"), this approach can be used to solve inequalities.

Using a Calculator or Computer Algebra System

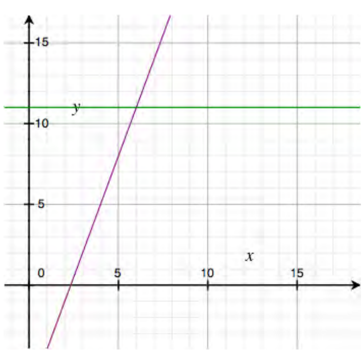
In CAS, it is possible to get the solution to the equation or inequality by just asking. There is not a lot to learn about algebraic manipulation that way, but it's a good way to solve equations when the focus is on applied or word problems, not on technical prowess and symbol sense. Some problems where the challenge is to find and set up the equation can be followed by a CAS solution, and a discussion of whether the solution makes sense.

The Cover-Up Method

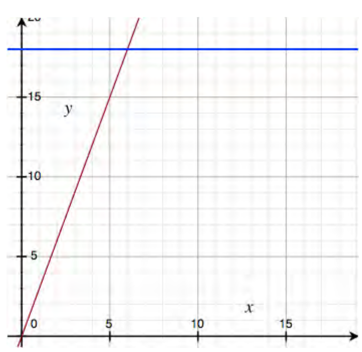
If *x* only appears once in an equation, even a fairly complicated one, it may be possible to use the "cover-up" method to repeatedly undo what was done to the *x*. This requires some sense of how to undo operations. An understanding of opposites and reciprocals comes in handy. See http://www.mathedpage.org/attc/cover-up.pdf.

Standard Algebraic Techniques

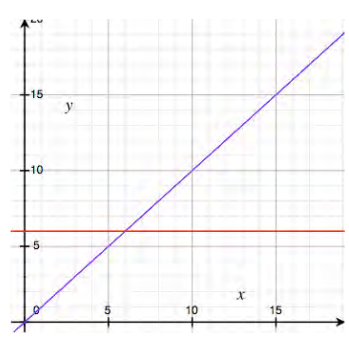
Given a traditional step-by-step solution of a linear equation, students are asked to explain each step. ("The same amount was added to both sides", etc.) They can also be asked to graph the two sides and see what the *x*-coordinate of the intersection is at each step. The fact that it doesn't change is an interesting confirmation that the equations are equivalent. Students can be paired, with one student taking the steps, and the other graphing the two sides as a check on the validity of the steps. For example, note that the *x*-coordinate of the intersection is 6 throughout in the sequence below, even though the equations are changing:



$3x - 7 = 11$



Add 7 to both sides: $3x = 18$



Divide both sides by 3: $x = 6$

Learning the standard algebraic techniques for inequalities is of course also possible, but it may not be a good use of time in this course, as the one difference from equation solving techniques is difficult to understand, and the availability of other approaches makes it less of a concern.

UNIT 6: NUMBERS AND GRAPHS

Overview

We review some big ideas of arithmetic with the help of graphs and graph paper. This includes the basics of signed number arithmetic, as well as the arithmetic of fractions. It is important to keep in mind that the purpose is not to get students to outdo calculators in accuracy or speed. It is to show that the rules of arithmetic actually make sense. The use of graphing and geometry helps make the unit more conceptual, and avoid the feeling that this is all elementary school material.

Time Estimate

3 weeks.

Key Concepts

Signed Number Arithmetic

Review the basic rules of addition, subtraction, multiplication, and division of signed numbers.

Fraction Arithmetic

Review the basic rules of addition, subtraction, multiplication, and division of fractions.

Introduction to Function Diagrams

See http://www.mathedpage.org/func-diag/. After a general introduction, Function diagrams are used in this unit to reveal properties of functions and throw light on algebraic notation.

Multiple Representations

The use of tables, graphs, function diagrams, and geometry to get insights into the operations of arithmetic serves as a reminder of the power of multiple representations.

Suggested Activities

Basic Function Diagrams

Students are asked to identify and discuss function diagrams illustrating a single operation, such as adding a number. See http://www.mathedpage.org/func-diag/pdfs/nine-fds.pdf. This activity illustrates properties of operations and algebraic notation.

Constant Sums

Ask for pairs of numbers that add up to a given positive sum. Arrange the answers in a table, and graph them as (x,y) pairs. The points lie on a line. Extending the line shows how negative numbers can be used as one of the numbers. Repeat with a sum of 0, and then with a negative sum. All those graphs are parallel lines with slope −1, and intercepts equal to the sum. Discuss those patterns, and especially what they tell us about adding positive and negative numbers. For example: "When x increases by 1, y decreases by 1. This pattern continues into negative numbers, so that when x is greater than the sum, y is negative to make up for that."

Giving "real world" examples of constant sum situations can motivate the activity. See some examples in *Algebra: Themes, Tools, Concepts* 5.1 and 13.1.

Constant Differences

Repeat the above activity with numbers whose difference is a given number D. Things proceed in a similar way, except that in this case we get a different graph if we are looking at $y - x = D$ vs. $x - y = D$. Discuss that, as well as slope, intercepts, and arithmetic patterns about subtracting signed numbers.

Constant Ratios

Repeat with numbers whose ratio is a given number. In this case, the graphs are lines through the origin, whose slope is the number, or its reciprocal (depending on whether we are looking at y/x or x/y). This ties in to the discussion of equivalent fractions (see Unit 4). Again, extending the lines gives us insights into signed number arithmetic. (For example, extending the line representing $y/x = 2$ into the third quadrant shows that the ratio of two negative numbers is positive.)

Using smaller and smaller positive numbers for y/x yields lines that are closer and closer to the x-axis. A ratio of 0 will give us the x-axis itself, since 0 times anything is 0. It is a small step from there to seeing that a negative ratio will give us a negative slope.

A constant ratio relationship is also known as a direct variation. See *Algebra: Themes, Tools, Concepts* 4.5, 4.6, and 4.B.

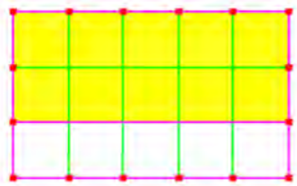
Constant Products

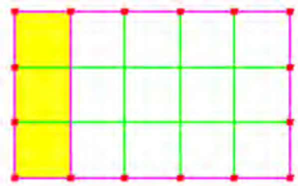
Repeat again, this time with numbers whose product is a given number. The graphs are no longer lines, and it is not easy to use those graphs to draw any conclusions about arithmetic. In order to get that payoff, one can use an electronic grapher and discuss the results. In addition to the conclusions one can draw about signed number arithmetic, it is also interesting and important to discuss how it is possible, for example, to multiply 100 by something and get 10 as the product. This helps dispel the misconception that "multiplying makes things bigger".

A "real-world" motivation is suggested in *Algebra: Themes, Tools, Concepts* 5.2.

Fraction Addition and Subtraction

Let's say we want to add 2/3 and 1/5. We will use grid paper to make it easier to visualize what is going on. If one unit was defined as a 3 by 5 rectangle, we could easily represent either fraction in it:

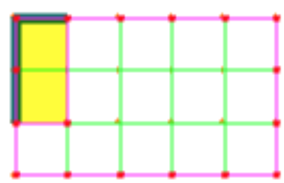




But now we see that the sum has to be 13/15. The same figure helps us see that the difference has to be 7/15. Students can decide on the appropriate rectangle for a given sum or difference. For a problem like 2/3 + 1/6, using a 3 by 6 rectangle would work, but it is more elegant (or saves paper, for the conservation-minded) to use a 1 by 6 rectangle, which will work for both fractions.

Fraction Multiplication and Division

For this, we'll use the area model for multiplication, which we first encountered in Unit 2. Let's say we'd like to multiply 2/3 by 1/5, but this time, we will represent them one-dimensionally using line segments on the sides of the rectangle. Again, a 3 by 5 rectangle will come in handy:



The product is 2/15.

See http://www.mathedpage.org/early-math/fractions.pdf for an approach to division, consistent with the above.

Unit 7: Growth and Change

Overview

The concept of rate of change, represented as slope and slope triangles on a Cartesian graph, was introduced in Units 3 and 4. We return to it here, in more depth, and use it to distinguish between linear and exponential growth and decay. This work about functions is connected to work about arithmetic and geometric sequences, which are perhaps more readily grasped. We also consider how rate of change plays out in quadratic functions. Finally, we build on students' understanding of slope to introduce the tangent ratio.

Time Estimate

3 weeks.

Key Concepts

Rate of Change

In order to go deeper than the well-known "rise over run" mantra, rate of change is reviewed as the ratio of change in the output over change in the input, and considered in representations other than the graph: tables and function diagrams.

Constant Rate of Change

The rate of change for linear functions, and only for linear functions, is constant. The rate of change of the rate of change for quadratic functions is constant. Functions that do not satisfy these conditions are neither linear nor quadratic.

Exponential Growth and Decay

In an exponential growth (or decay) situation, if you increase the input by a constant number, you multiply the output by a certain number (a growth factor). This is contrasted with the linear situation, where increasing the input by a constant number corresponds to a constant increase (or decrease) in the output. If the growth factor is less than 1, it is a case of exponential decay; if it is more than 1, we have exponential growth.

Arithmetic vs. Geometric Sequences

These share the properties, respectively, of linear and exponential functions, except that the input values (the subscripts in the usual notation) are natural numbers.

The Tangent Ratio

A solid understanding of the tangent ratio is attained through connections to the now familiar slope triangle (which we first saw in Unit 3) and similar right triangles (Unit 4). We do this work in the ten-centimeter circle (see www.mathedpage.org/circle/circles/10cm-circle.pdf). The tangent ratio is seen in two ways on the ten-centimeter circle: if we extend a line from the origin, making a certain angle with the positive *x*-axis, it will intersect the 10-cm circle, and also the vertical tangent to the circle through (10,0). Each of those intersections can provide a vertex of the slope triangle that starts at the origin, and each of them can be used to find the tangent of the angle. We introduce the arctangent of a number the same way. We will get acquainted with the sine and cosine in Unit 12.

Suggested Activities

Stairs on Graphs

Slope triangles can be added to any graph, and can help visualize the rate of change. This can be done both in abstract and "real-world" contexts, and for both linear and non-linear examples. See examples in *Algebra: Themes, Tools, Concepts* 8.1 and 11.3. See also a graphing calculator version for linear graphs:

http://www.mathedpage.org/calculator/stairs/.

Magnification in Function Diagrams

Function diagrams offer a different way to look at rate of change, which in this representation is sometimes called "magnification". The input–output lines for the function diagram of a linear function all meet in one point, called the focus. The position of the focus in relation to the axes determines the magnification. See http://www.mathedpage.org/func-diag/pdfs/focus.pdf.

Linear vs. Exponential Growth / Arithmetic vs. Geometric Sequences

A "real-world" example can be used to compare linear and exponential growth, and serves as well to introduce arithmetic and geometric sequences. In the first, a quantity increases by a constant amount during a given amount of time. In the other, it is multiplied by a given factor. Even if linear growth starts out faster, it is sooner or later overtaken and left behind by exponential growth. (See for example *Algebra: Themes, Tools, Concepts* 2.5, 8.7, and 8.B.) This is related to the difference between the question "how much more?" vs. "how many times as much?" (On this, see *Algebra: Themes, Tools, Concepts* 8.5, which will again be useful in Unit 10.)

Recognizing Functions

Students analyze tables of values for three types of functions (linear, quadratic, and exponential), and learn to recognize characteristics for each. For linear functions, the rate of change is constant, so that for a given increase in x, there is a corresponding increase in y. For quadratic functions, the rate of change changes linearly, so that if x values increase by a regular amount, y values will increase by values that change linearly (in other words, "second differences" are constant). Finally, in the exponential case, when you add a certain number to x, you multiply y by a certain number.

Once students master the three patterns, they should be able to identify functions of these three types by looking at tables of values.

Exponential Decay

The equation for exponential decay is the same as the one for exponential growth, except that the growth factor is less than 1. This can be modeled with dice, coins, or M&Ms. Starting with a given number of coins, say, each group of students flips all the coins, (e.g. by shaking a box,) and removes all the heads. They repeat this, keeping track of how many coins remain after each step. A class average is computed, which will almost certainly be approximated very closely by an exponential decay function, with growth factor ½. (See a version of this activity using 10-sided dice here: http://www.mathedpage.org/calculator/rolling/dice.pdf.) See also *Algebra: Themes, Tools, Concepts* 8.8.

Slope Angles

To motivate this activity, start by giving student one or two problems of the type described in "Using Slope Angles". The purpose of this activity is to build up the tools to solve such problems.

Students use the ten-centimeter circle to make tables of values based on measurements: For a given slope, what is the angle with the positive x-axis? For a given angle, what is the slope? At this point, it is not necessary to name the tangent ratio, but it is not a problem if some students recognize it. It is most convenient to measure the y-coordinate of the intersection with a tangent parallel to the y-axis through (10, 0), and divide by 10. (The activity may need to be preceded by a review of division by 10.) See *Geometry Labs* 11.1.

Using Slope Angles

Using the tables they created, students solve "real-world" problems that involve solving a right triangle with the help of the tangent ratio or the arctangent. See *Geometry Labs* 11.2. It is also possible to include problems that cannot be solved from those tables, but require additional work on the ten-centimeter circle to get the needed relationship between slope and angle.

Unit 8: Area and Distance

Overview

This unit is largely about geometry, but it also offers algebraic payoff all along the way. The opening discussion of distance helps set the stage, and is a way to introduce absolute value in a context where it is actually useful. The exploration of area formulas builds algebraic expressions on a geometric foundation. The work on geoboard area leads to the Pythagorean theorem. It involves many calculations and their generalizations into algebraic expressions, and it ends with a visual approach to simplifying radicals. While this approach has been abundantly tested with students, it is non-traditional, and therefore it is recommended that teachers take some time to work through the ideas.

Time Estimate

2 weeks.

Key Concepts

Squares and Square Roots

It is useful to approach the meaning of square roots in two ways. Geometrically, the square root of a number is the length of the side of a square that has that number for its area. Numerically, students should develop a sense of the approximate value of a square root by interpolating between known values. Any work with radicals that does not rest on those understandings is devoid of meaning. (We will look at the square root as a function in Unit 14.)

Taxicab vs. Euclidean Distance

Taxicab distance is the shortest distance from two points on a lattice if one is only allowed to travel horizontally and vertically. It is obtained by adding the horizontal and the vertical distances travelled. Of course, this involves the use of absolute value. Euclidean distance is the shortest distance "as the crow flies". In a Cartesian plane, it is obtained using the Pythagorean theorem (or equivalently but more opaquely, the distance formula.) In this unit, students will also learn another approach to Euclidean distance, based on squares and square roots.

The Pythagorean Theorem

In order to support the "$a^2 + b^2$" mantra many students are familiar with, it is helpful to base an understanding of the theorem on the geometry of area. This is of course the foundation of many proofs of the theorem. (Proofs using similar triangles are not appropriate to this course, as we did not include the AA similarity postulate.)

Suggested Activities

Area on Dot Paper

Find the area of various figures on dot paper. The key building blocks (in addition to rectangles) are right triangles with horizontal/vertical legs. Their area is half of a rectangle, and thus is readily found. By adding and subtracting those building blocks, one can find any area on dot paper. Of particular interest (for use later in the unit) are trapezoids and "tilted" squares. See *Geometry Labs* 8.4–8.5.

Area Formulas

The idea is for students to discover area formulas for different polygons, particularly different types of quadrilaterals. One way to do this is to add and subtract known areas as suggested in the previous activity. There are many ways to do this for each shape, and one decision that will have to be made is "a formula in terms of what?" (For example, in the case of a rhombus, there is a simple formula in terms of the diagonals.) Such an approach elevates the topic above mere memorization.

Square Roots and Distance

Define square roots, and estimate square roots of various numbers by comparing with known values. Give a geometric definition of square root, and apply it to find distance between any two points on dot: join the two points with a line segment; build a square that has this segment as one of its sides; find the area of the square; take the square root. If students want to use the Pythagorean theorem or distance formula, it is of course correct, but encourage them to also learn this approach, not because it is more efficient (it is not), but to deepen their understanding. See *Geometry Labs* 8.5, 9.1.

Taxicab Geometry

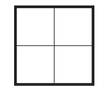
Start by comparing taxicab distance with Euclidean distance. Find a formula for taxicab distance. Explore some of the consequences of working with that metric. For example, what is a taxicab circle? See *Geometry Labs* 9.1, 9.6.

Confirming and Proving the Pythagorean Theorem

Generalizing the process of finding the area of a "tilted" square yields the Pythagorean theorem, which turns out to offer a shortcut to finding the distance. Each student makes a right triangle on dot paper, different from their neighbors', and draws a square on each side. Unless they make a mistake along the way, they will find that the sum of the areas of the squares on the legs is the area of the square on the hypotenuse. This makes the theorem concrete. See *Geometry Labs* 8.5, 9.2. It is also useful to see area-based proofs of the theorem. See for example http://www.mathedpage.org/constructions/pythagore/.

Extra: Simplifying Radicals

Simplifying radicals is no longer as important a skill as it once was, given the availability of electronic approaches to carry out this work rapidly and correctly. However simplifying radicals is helpful to facilitate communication, and it does provide a useful arena to develop both number sense and symbol sense. The usual approach to this topic, based on arithmetic, can be complemented with a geometric understanding based on area. For example: To simplify $\sqrt{8}$, we need the side of a square with area 8. If we divide this square into four equal smaller squares, each will have area 2. But then, looking at the side of the original square, we see that it is equal to $2\sqrt{2}$. Use this approach in parallel to throw light on the traditional arithmetic approach. Apply the techniques to finding the distances to the origin of all points with whole number coordinates from 0 to 10. See *Geometry Labs* 9.3, 9.4.



Extra: Pick's Formula

The area of a dot paper polygon (as long as its vertices are on lattice points) can be calculated from the number of inside dots and the number of boundary dots. Finding the formula for this is an interesting challenge. It helps reinforce understanding of area, while exploring a function of two variables. See *Geometry Labs* 8.6.

Unit 9: Trinomials and Identities

Overview

In this unit we return to the distributive law and factoring, which were addressed in Unit 2. This time we move into more challenging factoring problems, introduce the idea of factoring completely, and learn to appreciate the usefulness of some remarkable identities. We apply these techniques to solving some quadratic equations. This is an abstract and technical unit, whose main purpose is to increase student comfort with symbol manipulation, an important asset for those who will pursue mathematics or science beyond this course.

Time Estimate

2 weeks.

Key Concepts

Distributing and Factoring

We are mostly reviewing the distributive law and factoring, but we dig a little deeper by looking at more challenging examples. Recognizing common factors is key, as is an understanding of how to use trial and error to factor a trinomial, as it requires understanding the inverse relationship between distributing and factoring. One new idea is that while there are sometimes multiple ways to factor a polynomial, there is only one way to factor it completely.

Factoring Strategies

Factoring, as a skill, is not as important as it once was, given the availability of electronic help. However it is useful to develop one's ability to recognize algebraic patterns, and factoring provides a context to practice that. The most useful patterns here (besides recognizing a common factor) are the difference of squares and the squares of binomials.

The Zero Product Principle

It is a basic property of numbers that the product of nonzero numbers cannot be zero. (This is only true of zero. For example, the product of two numbers, neither of which is 1, can be 1.) Therefore if we know that the product of two expressions is zero, it follows that one or the other expression must be zero. This is helpful in solving certain equations.

Multiple Representations of Trinomials

Trinomials that can be factored over the integers can be represented geometrically using algebra tiles. They can be represented graphically by the graph of a parabola whose intercepts can readily inferred from the factored form. And finally, here is an interesting connection. If the leading coefficient of a trinomial is 1, and we can factor it, we have:

$$x^2 + bx + c = (x + p)(x + q)$$

with the sum of p and q equal to b, and the product equal to c. This can be represented nicely with the graphs of constant sum b and constant product c: the coordinates of their intersections are p and q.

Factored Form of Quadratic Functions

Special points on the graph of $y = a(x - p)(x - q)$ are the x-intercepts (p and q), the y-intercept (apq), and the vertex. The x-coordinate of the vertex can be obtained by averaging p and q. Substituting that value into the equation will yield its y-coordinate. The value of a determines the apparent shape of the parabola. A negative value implies a "frown" parabola and a positive value implies a "smile". For a given p and q, the y-intercept can be moved up or down by changing the value of a.

Suggested Activities

Square Window Panes

Assuming a large window is made up of smaller square panes, using three types: corner panes, edge panes, and inside panes. How many of each type are needed for different rectangular windows that use 72 panes? How many of each type are needed for different-sized square windows? To increase the side of an n by n window by a certain number m, how many additional panes are needed? See *Algebra: Themes, Tools, Concepts* 7.3.

Make a Rectangle

This is an activity using algebra tiles, where students use a given set of tiles to make a rectangle. We did this in Unit 2. This time we select problems that are conducive to seeing the underlying patterns. One way to do this is problems like these:

• Using one x^2, 12 units, and any number of xs, make as many different rectangles as possible.

• Using one x^2, 8 xs, and any number of units, make as many different rectangles as possible.

Problems of this type lead to the realization that if the trinomial can be factored into $(x + p)(x + q)$ then $p + q$ is the number of xs, and pq is the number of units in the original trinomial. Avoid minus in these problems.

Solving Equations by Factoring

After establishing the zero product principle, solve equations in the form $x(x + q) = 0$, $(x + p)(x + q) = 0$, $(x + q)(x + q)(x + r) = 0$, and $a(x + p)(x + q) = 0$. Then move on to solve equations of the type $x^2 + bx + c = 0$ by factoring.

Equations from Graphs

Students can find the factored form of the equation of a parabola from its intercepts, or from certain combinations of intercepts and vertex. For example, if the intercepts are (2,0), (4,0), and (0,4), p and q must be 2 and 4, and a must be ½. Likewise if one intercept is (2,0) and the vertex is (3,−1). Some of the examples should be of parabolas that are tangent to the x-axis.

Make a Square

Back to algebra tiles, this time to make squares. It is important to include problems of the type:

• Using one x^2, 8 xs, and any number of units, make a square.

• Using one x^2, 9 units, and any number of xs, make a square.

Such problems lead to understanding the pattern that is fundamental to completing the square. (We will return to this in Unit 13.) Examples of "Make a Square" puzzles can be found in *The Algebra Lab: High School* (see p. ix for a list).

Always, Sometimes, or Never True?

Open with the question: "Find two binomials whose product is a binomial". This should lead students to discover the difference of squares identity. Follow with problems like the ones in Unit 2, but this time also include examples based on the squares of binomials and the difference of squares.

Electronic graphing can be used to confirm answers. If the two sides of the equation yield the same graph, we have an identity (always true). If they yield graphs that intersect, the x values at the intersection(s) represent the solutions to the equation (sometimes true.) If the graphs never intersect, the statement is never true.

Extra: Factoring Trinomials and Graphing

If $x^2 + bx + c$ can be factored into $(x + p)(x + q)$, there are three interesting images that can be associated with it. One is the algebra tile factoring as in "Make a Rectangle". Another is the graph of $y = x^2 + bx + c$. It is interesting to discuss the intercepts of the parabola in relation to c, p, and q. And the third shows the graphs of $x + y = b$ and $xy = c$. Where those two intersect, we have points with coordinates whose sum is b, and whose product is c — in other words, p and q. See *Algebra: Themes, Tools, Concepts* 5.A, and http://www.mathedpage.org/parabolas/connections/index.html.

Unit 10: Powers and Roots

Overview

This unit, like Units 2 and 6, focuses on number sense, but maintains a high school flavor. The laws of exponents permeate the unit, and are reinforced by coming at them from three directions: number patterns, algebraic patterns, and "real-world" problems. This includes negative exponents. As for discussion and manipulation of fractional exponents, only the exponent ½ is considered, leading to the application of the laws of exponents to calculations involving radicals.

The unit builds on the work we did in Unit 7 about exponential growth and geometric sequences. We will review and extend this work in Unit 14.

Time Estimate

2 weeks.

Key Concepts

The Laws of Exponents

The laws of exponents are surprisingly difficult to learn for many students. The approach we take, consistent with the rest of this course, is to aim for an understanding of a rationale for the laws, with the basic law ($x^a \cdot x^b = x^{a+b}$) a foundation for the rest. In other words, the rules are rules not because the book or the teacher says so, but because they make sense. Understanding that helps students remember the rules, or at least equips them to re-discover them if necessary.

Zero and Negative Exponents

Negative exponents are a logical consequence of the laws of exponents. Moreover, one can make sense of them by reflecting on exponential growth and decay. For example, if a population of bacteria doubles every hour, and the population was 200 at a certain time, what was it two hours before?

The Exponent ½

The meaning of the exponent ½ also follows logically from the laws of exponents, and it too can be thought about in an applied context, by way of the concept of the halfway growth factor. For example, if a population of bacteria grows by a factor of 9 every hour, what is the growth factor for every half-hour?

Extra: Working with Radicals

Doing arithmetic with radicals involves the laws of exponents, handling expressions like $\sqrt{a} \cdot \sqrt{b}$, and the distributive law for $p\sqrt{a} + q\sqrt{a}$. Understanding why the square root of a number less than 1 is greater than the number, and why negative numbers do not have a real number square root, forces a review of basic multiplication. While rationalizing the denominator is not as important of a skill as it used to be, it does provide a worthwhile context for applying the rules, and it does facilitate communication (for example by allowing one to see that $2/\sqrt{3}$ represents the same quantity as $2\sqrt{3}/3$.)

Suggested Activities

A Model for Population Growth

Examine a model for bacterial population growth where population doubles (or triples) every hour, and answer questions like "The number of bacteria after 10 hours is how many times as big as it was after 7 hours?" This leads to seeing that $2^7 \cdot 2^3 = 2^{10}$, and of course, the corresponding divisions. Representing such multiplications as follows should reinforce this understanding:

$$(2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2)(2 \cdot 2 \cdot 2) = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2.$$

See *Algebra: Themes, Tools, Concepts* 8.5, 8.6.

Equal Powers

Using only whole number powers greater than 1, students answer questions such as "Find powers of 2 that can be written as powers of 8", or "Write 27^3 using a different base". This puzzle-like approach leads to a generalization about the power of a power. See *Algebra: Themes, Tools, Concepts* 8.9.

Working with Monomials

By exploring the multiplication and division of monomials, students generalize the laws of exponents. For example: "Write x^5 in three ways as a product of monomials," or "Write $(5x^3)(5x^3)$ as a square, then write it without parentheses." This, in combination with the previous two activities, leads to stating all the laws of exponents. See *Algebra: Themes, Tools, Concepts* 8.10.

Zero Exponents, Negative Exponents

By extending number patterns (2^3, 2^2, 2^1, ...) students can derive the meaning of negative exponents. They can confirm their discovery by looking at applying the laws of exponents to such expressions as $2^2 \cdot 2^{-2}$ and $2^2 \cdot 2^{-3}$. This provides an opportunity to explore ratios of powers further. It is also a good time to compare expressions such as $(-5)^4$ and -5^4. See *Algebra: Themes, Tools, Concepts* 7.8 and 8.11.

The Halfway Growth Factor

The laws of exponents can also be used to throw light on the meaning of the exponent ½ by comparing $a^{½} \cdot a^{½}$ with $\sqrt{a} \cdot \sqrt{a}$. The model for population growth discussed above provides another angle to think about this: if the growth factor for an hour is a, what is the growth factor for a half-hour? Trial and error quickly shows that the answer must be $\sqrt{a}$. See *Algebra: Themes, Tools, Concepts* 9.7, 9.8.

Extra: Radical Operations

Applying the distributive law and the laws of exponents to operations with radicals. The geometric meaning of square root and simplifying radicals was a topic in Unit 8, and can be reviewed here. This is also a good time to see what happens with $\sqrt{a}$ when $a < 1$. See *Algebra: Themes, Tools, Concepts* 9.4, and the opening of 9.5.

UNIT 11: SATISFYING CONSTRAINTS

Overview

The unit is largely about solving systems of linear equations and inequalities. Once again, as we did in Unit 5 for solving equations in one variable, we stress that there are many ways to do this. Along the way we will learn about different algebraic representations of linear functions. We will also learn about how this relates to systems of inequalities, building on the work from Unit 3. Finally, we will apply all this to linear programming, an interesting "real-world" application, where we maximize a certain quantity given a number of constraints.

Time Estimate

2 weeks.

Key Concepts

Solving Systems of Equations: What Does It Mean?

As always, it is important for students to understand what a given process is meant to accomplish, if we expect the process to make sense to them. Specifically, prior to learning techniques for the solving of systems, they should understand the goal. One way to see this is that we are looking for values of two variables that satisfy two equality constraints. Typically, imposing just one such constraint yields an infinite number of solutions, but adding another constraint can narrow this down to one. This can be visualized with the help of graphs.

Multiple Algebraic Forms for Linear Functions

Consider the equation $ax + by = c$. Learn to get from this form to slope-intercept form by solving for y. (This is not always possible! Students should understand why it is impossible when $b=0$.) Be able to find slope and intercepts directly, without going to $y = mx + b$. Recognize that in this form, we can write the equation for any line, including vertical lines.

One can get to the equation of a line from the coordinates of two points on the line, or from one point and the slope, by solving a system of linear equations. Memorizing point-point and point-slope forms without that understanding is not likely to succeed.

Solving Linear Systems: Different Strategies

As in the case of equations in one variable, it is possible to use CAS to solve systems. Another way is to look for the intersections of graphs. (Both of those approaches can work whether the equations are linear or not.) In addition to familiarity with electronic approaches, it is useful to develop some facility with algebraic approaches: substitution and linear combinations. The goal is to develop symbol sense, not so much to match the machines in speed and accuracy. While it is traditional and helpful to know how to solve systems where both equations are given in standard form, it is important to be able to work from any form.

Extra: Graphing Inequalities in Two Variables

As we saw in Unit 3, the graph of most of the functions we see in this course divides the plane into two regions: above the graph, y is greater than the function, and beneath it, it is less. This helps to visualize inequalities in two variables. For example, say that $2x + 3y < 9$. One can solve this for y, and use the method from Unit 3. Or, one can graph $2x + 3y = 9$, and test a point (for example the origin) to determine which side of the graph is the one where the inequality is true. In this case, $0 < 9$, so the inequality is true in the region beneath the graph.

Extra: Linear Programming

Linear programming is a "real world" application of inequalities and systems of equations. Given some inequality constraints on the relationship between two variables, each constraint is graphed. The region of the plane where all the inequalities are true is called the *feasible region*. It turns out that another variable that depends linearly on the two variables will be maximized (or minimized) at one of the vertices of the feasible region.

Suggested Activities

From One Constraint to Two

Students answer an open-ended question such as "Find numbers x and y such that $x + 2y = 12$". Answers can be organized into a table, and graphed. However, adding another condition, such as "x and y are opposites" gives us a unique solution. See *Algebra: Themes, Tools, Concepts* 10.3.

Standard Form

Generalize the "Constant Sums" equations from Unit 6 to $ax + by = c$. To get a sense of what the graph of this looks like, compare the intercepts of $x + y = c$ with the intercepts of $ax + by = c$ (first in specific cases before looking at the generalization using the parameters.) Use the intercepts to quickly find the slope. What happens if $a = 0$? $b = 0$? How does this affect slope and intercepts? Learn to convert the equation into slope-intercept form. When is that not possible? See *Algebra: Themes, Tools, Concepts* 10.5.

Solve "Real-World" Systems

Translate traditional two-variable word problems into equations, and solve the systems electronically. One the one hand, electronic solution makes it possible to emphasize the translation of words into equations rather than the technicalities of the solving process. On the other hand, a graphical approach throws additional light on the problem, including those cases where there are no solutions or an infinite number of solutions. See, for example, *Algebra: Themes, Tools, Concepts* 10.7.

Use Algebra to Solve Linear Systems

Learn to use substitution, linear combinations, and basic algebra to solve linear systems. The Seeing Math System Solver at http://seeingmath.concord.org/sms_interactives.html is a great way to work on this. For example, start with a given system on the projector, and ask students for suggestions on how to solve it. All the usual solving tools are available, and at each step the Solver gives you the updated equations — with no distracting mistakes. Using the slider on the right yields a graph of the current system. You continue until that graph consists of a horizontal and vertical line! (Or, as one teacher put it, "add till it's plaid".) See also http://blog.mathedpage.org/2011/04/add-till-its-plaid.html if you have some other way to graph $ax + by = c$.

Point-Point and Point-Slope

Apply equation-solving techniques to find the equation of a line given two points, or given a point and the slope. In particular, you can use the Seeing Math Linear Transformer at http://seeingmath.concord.org/sms_interactives.html to explore point-slope form.

Extra: The Feasible Region

Inequalities in two variables can be graphed: one side of the boundary curve will satisfy the inequality, and the other will not. The curve itself is included in the feasible region if we have a "greater than or equal" situation. Graphing the boundary curve can be tricky, so it is best to do this with the help of CAS, or with electronic graphers that do not require solving for y first. If we are looking at more than one inequality, the region of the plane where the solutions reside is the intersection of the regions (assuming all the inequalities need to be true at the same time).

Extra: Linear Programming

Start with a "real-world" problem involving a set of linear inequality constraints in two variables. Translate those into inequalities in x and y, and find the feasible region. A quantity that depends linearly on x and y will be maximized and minimized at vertices of the feasible region. See for example http://www.mathedpage.org/alg-2/letters-postcards.pdf.

Unit 12: Angles and Ratios

Overview

This is a review of basic trigonometry, tying together various concepts we have encountered earlier in the course, and breaking some new ground. Once again, the ten-centimeter circle is a key tool for this work, as it offers a concrete environment and an introduction to circle trigonometry, and moreover helps ground the rote universe of soh-cah-toa. Using the Pythagorean theorem and the tangent ratio, students can solve most right triangles, using what was learned in Unit 7. Adding the sine and cosine to our repertoire unlocks all right triangles. The unit ends with applications of the sine and cosine, and the graphs of the main three trig functions.

Time Estimate

2 weeks.

Key Concepts

Solving Right Triangles

Students learn that two pieces of information can be sufficient to solve a right triangle. If we have a leg and an angle, we can use the tangent ratio to get the other leg. If we have two sides, we can get the third with the Pythagorean theorem, and the angles with the help of the arctangent. If we have the hypotenuse and an angle, we must resort to sine or cosine. If we only have an angle, we cannot solve for the sides, though we can figure out what the ratio of the legs will be.

The Sine and Cosine

These ratios involving the hypotenuse are motivated by how much they facilitate solving certain right triangles. They are introduced on the ten-centimeter circle in order to make it easier to use measurement to ground the definition, and to facilitate the transition to angles greater than 90°. The arcsine and arccosine functions are also introduced, but only for angles in the first quadrant.

The Basic Trig Functions

Students should learn to recognize key features of the graphs of the sine, cosine, and tangent functions. Even though we used the inverse trig functions in solving right triangles, we will not study them as functions in this course.

Suggested Activities

Solving Right Triangles

Students solve right triangles given two bits of information, one of which is a leg. If no side is given, no side can be found. If only the hypotenuse is given, the problem is quite challenging. See *Geometry Labs* 11.3.

Ratios Involving the Hypotenuse

The ten-centimeter circle is used to make a table of sine and cosine and their inverses, for angles between 0° and 90°. See *Geometry Labs* 11.4. The sine and cosine for angles outside of this range will be defined in the graphing activity below.

Using Triangle Trig and the Pythagorean Theorem

Introduce the terminology of tangent, sine, and cosine. Point out that the tangent of an angle equals its sine divided by its cosine. Using the ten-centimeter circle, tables they created, a calculator, and/or the Pythagorean theorem, students solve "real-world" problems. See for example *Geometry Labs* 11.5, or almost any textbook.

Famous Right Triangles

Half-square, half-equilateral, 1-2-$\sqrt{5}$, 3-4-5, and 5-12-13 triangles offer a chance to review the Pythagorean theorem, and to apply trigonometry. Memorization of the sides and angles should be grounded in understanding, not a substitute for it. See *Geometry Labs* 10.7.

Graphing the Trig Functions

Define the unit circle (possibly using one decimeter as the unit). Define sine and cosine as the coordinates of a point on the unit circle. Find the sines and cosines of various special points around the circle, using famous right triangles in the first quadrant as a reference. Graph the trig functions, and discuss special points on the graph.

Unit 13: Parabolas and Quadratics

Overview

Once again, as we did in Unit 9, we use the quadratic domain to deepen students' symbol sense and comfort with symbol manipulation. We focus on new symbolic representations of quadratic functions, and their connections with the graphs of parabolas. We build an understanding of standard form on the work we've already done on factored form. We connect vertex form to the broader idea of transformations: the shifting and stretching of graphs. Finally, we introduce the quadratic formula on a foundation of completing the square with algebra tiles.

Time Estimate

2 weeks.

Key Concepts

Quadratic Forms

Get to know the three forms of quadratic functions (standard, vertex, and factored when possible.) Use algebraic manipulation to explore the connections between them, and how the various parameters are related to the graph. Note that the parameter a in all three forms affects the apparent shape of the parabola, and previews vertical stretches.

Standard form ($y = ax^2 + bx + c$) makes the y-intercept obvious — if x is 0, y is c. Vertex form can be introduced without needing to complete the square, and serves as a first example of horizontal and vertical shifts.

Extra: Algebraic Connections

The x-coordinate of the vertex can be deduced from the values of a and b. To prove this, we connect this form (standard form) with factored form after distributing, and use that to see what the sum of the x-intercepts must be if they exist. Knowing their sum, we can easily get their average. Another approach, also grounded in the symmetry of the parabola, yields the same answer if there are no roots. (Rewrite standard form as $y = x(ax + b) + c$, and find the two points with y-coordinate equal to c. The x-coordinate of the vertex must be halfway between them.)

Equal Squares and Completing the Square

Quadratic equations that are presented as an equality of squares are readily converted to a pair of linear equations, and are thus not so difficult to solve. This leads to the strategy of completing the square to solve quadratic equations. We do this on the basis of some of the groundwork we did using algebra tiles in Unit 9. Note that we can also make a connection between the graph of a quadratic function and completing the square.

The Quadratic Formula

Once students understand completing the square, they can use that method to solve any quadratic. Using it in the general case yields the quadratic formula. The formula can be used to confirm some of the results we obtained when analyzing $y = ax^2 + bx + c$ (the sum of the roots, the x-coordinate of the vertex).

Suggested Activities

Maximizing Area

What is the maximum rectangular area you can enclose if you have a certain length of fencing available? See http://www.mathedpage.org/attc/max-area-condensed.pdf. Part of the purpose of the activity is to review factored form, and introduce the vertex and the symmetry of the graph.

Extra: From Factored to Standard Form

Starting from what students know about the factored form of quadratic functions, derive some key ideas about standard form. This involves a lot of symbol manipulation. See above, and http://www.mathedpage.org/parabolas/forms.pdf.

Vertex Form and Moving Graphs

Establish vertex form with the concise approach used in the above link, and generalize from there to the shifts of any graph. Specifically: replacing x by $(x - h)$ moves the graph horizontally by h units; replacing y by $(y - v)$ moves the graph vertically by v units.

Extra: Stretching and Flipping Graphs

Use the Concord Consortium's Seeing Math Quadratic Transformer (http://seeingmath.concord.org/sms_interactives.html) to explore horizontal and vertical shifts, reflections, and stretches using all three forms of a quadratic function. Generalize to other graphs such as perhaps absolute value graphs, or cubics. End with a "Make These Designs" challenge.

Equal Squares

Solve equations in the form $x^2 = 25$, or $(x - 4)^2 = 3$. The key is to realize that there are two solutions for each, since (in the latter case) both $x - 4 = 3$ and $x - 4 = -3$ lead to correct answers. See *Algebra: Themes, Tools, Concepts* 7.7, 13.6.

Completing the Square

Rearrange equations to get them into the equal squares form. It is helpful to start this using the area model of algebra tile squares we saw in Unit 9, but the activity should end with examples that cannot be done that way. See *Algebra: Themes, Tools, Concepts* 13.6. See also http://www.mathedpage.org/parabolas/completing/.

Extra: Maximizing Volume

Make a box (without a top) by cutting out equal squares from the corners off a rectangular piece of cardboard. How big should the squares be in order to get the maximum volume? Students will need to use an electronic grapher to solve this. The idea is to show that a cubic does not have the same symmetry as a parabola, and that one needs more math to solve this problem (calculus, in fact). See *Algebra: Themes, Tools, Concepts* 13.5.

UNIT 14: USING EXPONENTS

Overview

We apply the laws of exponents to develop an understanding of exponential functions, compound interest, scientific notation, and logarithms.

Time Estimate

2 weeks.

Key Concepts

Real Number Exponents

Other rational exponents follow a similar pattern to the exponent ½, but manipulating them is probably not a priority for this course. However, a full understanding of exponential growth requires some grasp of what happens between whole number values of the input, so we look at real number exponents and the concept of interpolation, using technology.

Compound Interest

We use this "real-world" concept to review the distributive law and exponential growth.

Scientific Notation

We learn how integer powers of ten are used to represent very large and very small numbers. This provides a useful context to practice laws of exponents, as well as review the arithmetic of multiplying and dividing by powers of ten.

Extra: Square Roots and Absolute Value

Except for the work on solving quadratics (equal squares and the quadratic formula) we have mostly worked with square roots in relation to distance and the Pythagorean theorem. These are contexts where only one of the square roots (the positive one) is relevant, and only known constants are under the radical. Things get more complicated when we consider expressions of the type $\sqrt{x^2}$. Using the square root as a function forces this expression to represent a single output: whether x is positive or negative, the output is $|x|$.

Another benefit of looking at the square root as a function is that graphing it along with the $y = x$ line clearly shows that when $0 < x < 1$, we have $\sqrt{x} > x$. This should also be confirmed by using examples, with x represented as a fraction and as a decimal.

Extra: Logarithms

We offer an introduction to the basic properties of logarithms, building upon the foundation of scientific notation and the laws of exponents. Then we use those properties to solve some equations involving exponents.

Suggested Activities

Graphing Powers

Up to this point, we have mostly built our understanding of exponents on the foundation of exponentiation as repeated multiplication. However, graphing $y = 2^x$ (or some other exponential function) on an electronic grapher shows that there must be a meaning to expressions where the exponent is not a whole number. While a full understanding of what is happening is beyond what can be accomplished in this course, one way to glimpse at what is going on is to explore the meaning of an expression such as $2^{3.5}$. Using the laws of exponents, we see it must be equal to $8\sqrt{2}$, and sure enough, that is what a calculator shows. This is a good opportunity to compare the graphs of $y = 2^x$, $y = 2^{-x}$, $y = 0.5^x$, and $y = x^{.5}$.

Savings Account

Our exploration of compound interest starts with building a table, perhaps using a spreadsheet. In order to get a formula for what happens after t years, we need to convert percents to decimals, and use factoring to convert expressions like $b + .01b$ to $1.01b$.

Very Large and Very Small Numbers

Using scientific notation makes it possible to manipulate very large and very small numbers. To develop an understanding of this, we introduce that format with powers of 2. Doing arithmetic with numbers in scientific notation reviews the laws of exponents, negative exponents, and the shortcuts for multiplying and dividing by powers of ten. Interesting applications of scientific notation include the scale of the solar system, geological time, the scale of the universe, and the elementary particles that make up everything. See *Algebra: Themes, Tools, Concepts* 7.9, 7.10, 7.11, 8.10, 8.12, and http://htwins.net/scale2/.

Extra: Super-Scientific Notation

This is a lesson that builds on scientific notation, by using technology to find the power of 10 that corresponds to a given number. Manipulating those exponents is a straightforward application of the laws of exponents, but it leads directly to the concept of logarithm. See http://www.mathedpage.org/calculator/#super.

Extra: Solving Exponential Equations

Using logarithms to solve equations of the type: $3^{2x} = 99$.

Extra: Graphing $y = \sqrt{x^2}$

This is an activity to unpack the challenging idea that $\sqrt{x^2} = |x|$. It is best done with the help of an electronic grapher. See http://www.mathedpage.org/calculator/#sqrt.

Unit 15: Functions and Modeling

Overview

This is largely a review unit, so there is little in the way of new concepts. We review some highlights of the course: different types of functions, similar figures, arithmetic and geometric sequences, and linear and exponential growth.

Time Estimate

2 weeks.

Key Concepts

Nth Power Variation

In Unit 6, students learned to recognize direct variation ($y = kx$) with its constant ratio pattern. In Unit 7, students learned to recognize numerical patterns in the table of values for linear, exponential, and quadratic functions. In this unit, we review those, and add a little more. For n^{th} power variations ($y = kx^n$), we have a "multiply-multiply" pattern: when we multiply an x value by a certain number, we multiply the corresponding y value by that number raised to the n^{th} power. In the particular case of an inverse variation ($y = k/x$), we have a constant product pattern.

π

Students discover the relationship of the circumference and area of a circle to its radius.

Iterating Functions

To iterate a function means to take its output as the next input, and repeat the process. Iterating linear functions provides an interesting type of linear or exponential change, and in particular cases, of arithmetic and geometric sequences.

Suggested Activities

What's My Rule?

After introducing the patterns for n^{th} power variation and inverse variation, tables are given in mixed order for linear, quadratic, exponential, direct variation, inverse variation, and n^{th} power variation. For each, students must recognize the pattern and find the formula.

Area and Dilation

Students use interactive geometry software to make tables of values for the areas and perimeters of similar figures obtained with the dilation tool. They find formulas for the perimeter and area as functions of the scaling factor. They also use the software to make tables and find formulas for the perimeter and area of circles as functions of their radius. The latter problems of course, lead to the number π.

Perspective

This is a lab where students measure the apparent height of another student when they stand at different distances. The apparent height is obtained by holding a yardstick at arm's length. If the measurements are done carefully, the table of results is close to a constant product situation, and thus an inverse variation. Students can use similar right triangles to confirm that the pattern makes sense. See www.mathedpage.org/alg-2/perspective.pdf.

US Population Growth

This is an opportunity to apply linear and exponential growth models to the growth of the US population in the period covered by the Census (1890 to 2010.) The models can be used to make predictions and to interpolate. See *Algebra: Themes, Tools, Concepts* 12.1.

Mathematical Models in Science and Society

Some possibilities are: a lesson based on a formula for blood alcohol concentration, and how it decreases over time (public health); a lesson based on Charles's Law (chemistry); a lesson based on the behavior of springs (physics). See *Algebra: Themes, Tools, Concepts* 12.3, 12.4.

Extra: Iterating Linear Functions

This is a rich activity (or more than one activity), which ties in to arithmetic and geometric sequences, and serves to review them as particular cases of it. Various "real-world" phenomena can be modeled this way, and an excellent way to understand what happens is with the help of linked function diagrams. See http://www.mathedpage.org/iterating/.