A Different Approach to the Quadratic Formula

This is an approach to the quadratic formula based on “constant sums, constant products”.

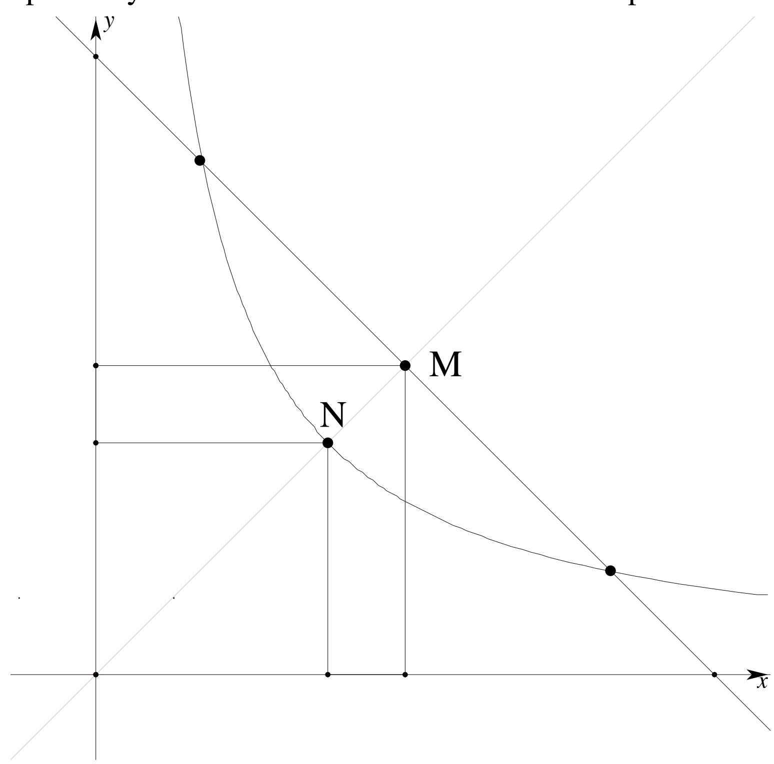
Say that two numbers p and q have a sum equal to $-b/a$ and a product equal to c/a (with $a \neq 0$).

1. Show that p and q must be solutions to the quadratic equation $ax^2 + bx + c = 0$. (Hint: consider the system consisting of the sum and product equations; eliminate p or q.)

2. *In other words, if we solve the system $xy = c/a$ and $x+y = -b/a$, we have solved the corresponding quadratic equatio*n. (p, q) and (q, p) are the points of intersection of the graphs in the figure below. Label the graphs and points in the figure to make this clear. (Say $p>q$.)

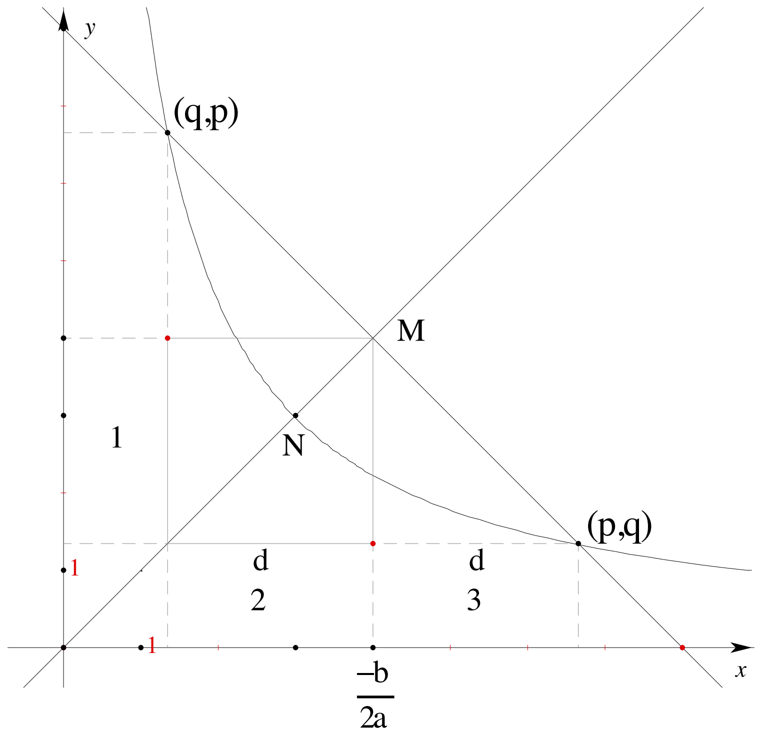
3. Sketch a quick graph for each of these two cases:
   a. If c/a is negative, there has to be an intersection. Why?
   b. If c/a is positive, there may or may not be an intersection. Why?

4. Study the figure. The graph of $y=x$ intersects the constant sum and constant product graphs at points M and N respectively. Find the coordinates of M and N. Explain.



5. Consider the square with opposite vertices at the origin and M, and the square with opposite vertices at the origin and N. Call the area of the first square, minus the area of the second square Δ. Shade that difference area on the figure. *The constant sum and constant product graphs will intersect if Δ is a positive number or zero*. Why?

6. Write the italicized sentence from #5 algebraically. Simplify. *Use this to rediscover the discriminant*.



7. Study the figure. On the x-axis, p and q are on either side of –b/2a, at the same distance d from it. Why?

8. On a different set of axes, sketch the graph of y= $ax^2 + bx + c$, in a case where the parabola that crosses the x-axis. Label that graph with p, q, –b/2a and d in the appropriate locations.

We have $p = \frac{-b}{2a} + d$ and $q = \frac{-b}{2a} - d$, or in a single equation: $x = \frac{-b}{2a} \pm d$. *If we find d in terms of a, b, and c, we have a formula to solve all quadratic equations*. To find d, we will use this fact: d is the side of a square. d^2 is the area of the square. (The medium-sized square in the figure.)

9. Explain this (you may use color to follow the argument on the figure—no algebraic calculations are needed):
   d^2 = area of the large square – (area 1 + area of the small square + area 2)
   = area of the large square – (area 3 + area of the small square + area 2) (Why?)

10. Reminder: the curve is an equal products graph.
    d^2 = area of the large square – area of the square with opposite vertices at the origin and N (Why?)
    = Δ, the area we calculated in problem #6

11. Use what you learned in this lesson to write p and q in terms of a, b, and c, (or to write a single equation for x in terms of a, b, and c) i.e. to rediscover the quadratic formula.

Notes on "A Different Approach"

This is a lesson for teachers, though it is not out of the question to use it with students in a precalculus class.

I suggest you print (or duplicate) the lesson on two separate sheets of paper rather than back-to-back or on stapled sheets, so that the user can see both figures at once.

Clarify before you start that this is another approach to familiar material (the discriminant on page 1, the quadratic formula on page 2)—the lesson includes no new results.

Prerequisites

This lesson assumes familiarity with the graphs of Constant Sums and Constant Products. More info on these can be found at:

www.MathEducationPage.org/constant/

Another important background ingredient is familiarity with the discriminant, the quadratic formula, the sum and product of the roots of a quadratic, and the axis of symmetry of a parabola.

Some teachers (and all the more so students) would benefit from working through a numerical example such as $-b/a = 7/2$, and $c/a = 15/2$ before tackling the symbolic approach in these worksheets.

Comments

1. If p and q are the solutions of the quadratic $y= ax^2 + bx + c$, then their sum is $-b/a$ and their product is c/a. This can be shown either with the quadratic formula, or by distributing $y=a(x-p)(x-q)$. This problem is about establishing the converse of this result.

3. Note that c/a is negative whenever ac is negative, and draw a connection with the sign of the discriminant when a and c have opposite signs.

4. $M\left(\frac{-b}{2a},\frac{-b}{2a}\right)$; $N\left(\sqrt{\frac{c}{a}},\sqrt{\frac{c}{a}}\right)$

6. $\frac{b^2-4ac}{4a^2}$ is non-negative whenever the numerator is non-negative, since $4a^2$ is always positive.

7. Use geometry, or the fact that $-b/2a$ is the average of p and q.

10. This is the subtlest step in this proof, and in fact the heart of the proof.

Making the connections between this approach and others makes for a worthwhile discussion.